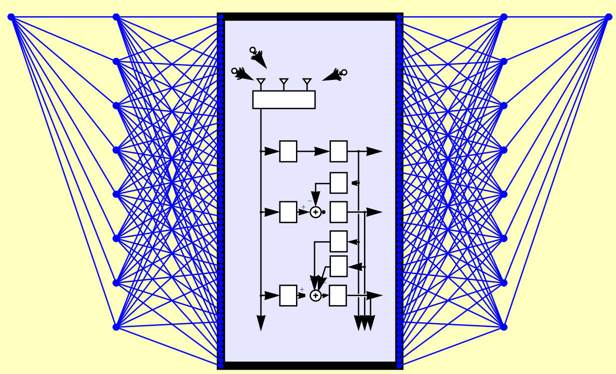
Solutions Manual

for

Digital
COMMUNICATION
third edition



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Chapter 1

Problem 1-1.

(a) The overload point of the A/D converter (largest signal that can be accommodated) will be chosen on the basis of the signal statistics and the signal power so as to keep the probability of overload low. Assuming the signal doesn't change, we would want to keep the overload fixed. Hence, the Δ would be halved.

(b) Generally the error signal would be halved in amplitude. This would increase the SNR by $20\log_{10}2 = 6$ dB.

(c) The bit rate would increase by f_s, the sampling rate.

(d) We get, for some constant K,

$$SNR_{\text{dB}} = 6n + K \ , \qquad f_b = nf_s \ , \tag{S.1}$$

and thus

$$SNR_{\text{dB}} = \frac{6f_b}{f_s} + K \ . \tag{S.2}$$

In particular, the SNR in dB increases linearly with the bit rate.

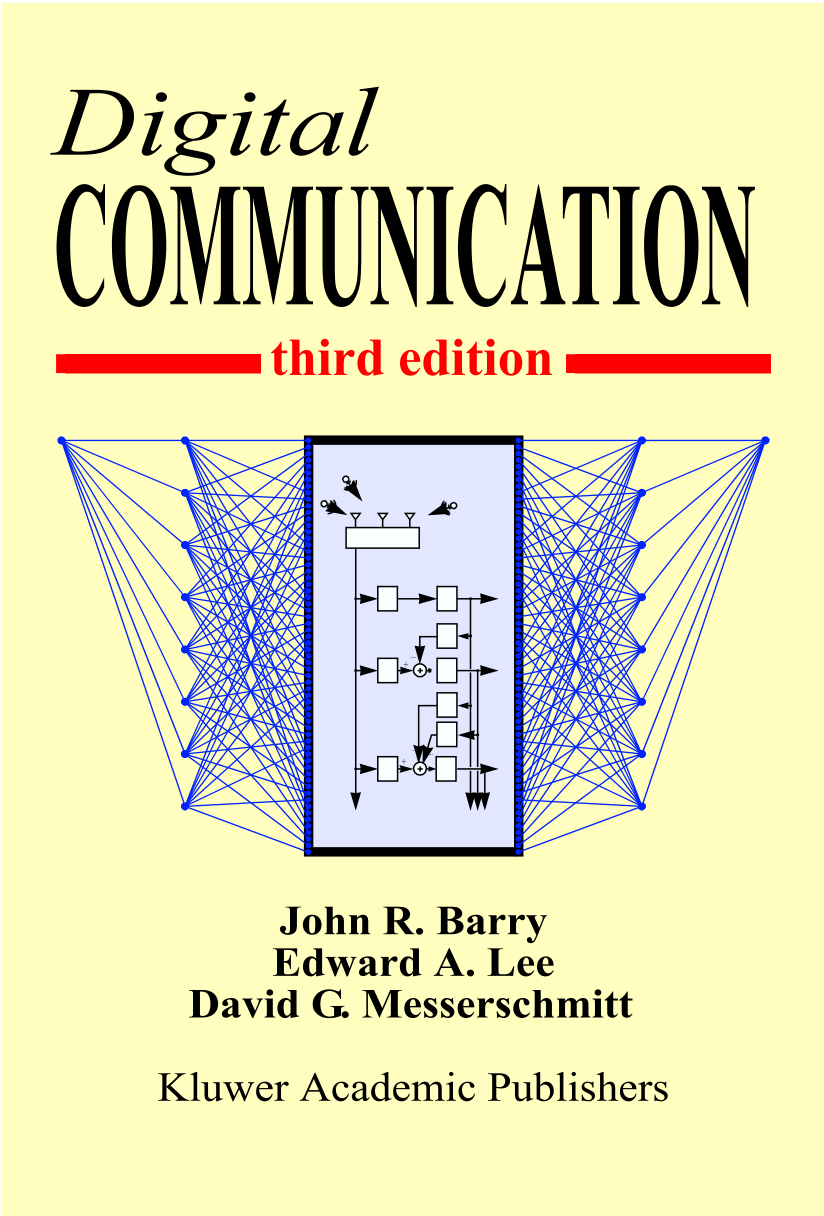
Problem 1-2. Each bit error will cause one recovered sample to be the wrong amplitude, which is similar to an added impulse to the signal. This will be perceived as a "pop" or a "click." The size of this impulse will depend on which of the n bits of a particular sample is in error. The error will range from the smallest quantization interval (the least-significant bit in error) to the entire range of signal levels (the sign bit in error).

Problem 1-3. The most significant sources will be the anti-aliasing and reconstruction lowpass filters, which will have some group delay, and the propagation delay on the communication medium. Any multiplexes (Chapter 17) will introduce a small amount of delay, as will digital switches (Chapter 17).

Problem 1-4. Assume the constant bit rate is larger than the peak bit rate of the source. Then we might artificially increase the bit rate of the source up until it precisely equals the bit rate of the link by adding extra bits. We must have some way of identifying these extra bits at the receiver so that they can be removed. A number of schemes are possible, so here is but one: Divide the source bits in to groups called packets with arbitrary length. Append a unique sequence of eight bits, called a flag, to the beginning and end of each packet, and transmit these packets on the link interspersed with an idle code (say all zeros). The only problem now is to insure that the flag does not occur in the input bit stream. This can be accomplished using line coding techniques. For example, the flag might be a sequence of 8 consecutive ones, and the transmitter might use the Manchester code, sending 01 to convey a 0 bit, and 10 to convey a 1 bit.

Solutions Manual

for



Chapter 2

Problem 2-1. We start out with an easy problem! Looking at Fig. 2-2, when the imaginary part of the impulse response is zero, we see that the system consists of two independent filters, one for real part and one for imaginary part of the input, with no crosstalk. The imaginary part of the impulse response results in crosstalk between the real and imaginary parts.

Problem 2-2. Doing the discrete-time part only, write the convolution sum when the input is $e^{j2\pi fkT}$ as:

$$\begin{aligned} y_k &= h_k * e^{j2\pi fkT} \\ &= \sum_{n=-\infty}^{\infty} h_n e^{j2\pi f(k-n)T} \\ &= e^{j2\pi fkT} \sum_{n=-\infty}^{\infty} h_n e^{-j2\pi fnT}. \end{aligned} \tag{S.3}$$

The output is the same complex exponential multiplied by a sum that is a function of the impulse response of the system h_n and the frequency f of the input, but is not a function of the time index k. This frequency response

$$H(e^{j2\pi fT}) = \sum_{n=-\infty}^{\infty} h_n e^{-j2\pi fnT} \tag{S.4}$$

is recognized as the Fourier transform of the discrete-time signal h_n.

Problem 2-3.

(a) Let $z(t) = x(t) * g(t)$, which has Fourier transform:

$$Z(f) = X(f)G(f), \tag{S.5}$$

Since $w_k = h_k * z(kT)$, it follows from (2.17) that

$$W(e^{j2\pi fT}) = H(e^{j2\pi fT}) \frac{1}{T} \sum_{m=-\infty}^{\infty} X(f - m/T)G(f - m/T) \tag{S.6}$$

The output signal Fourier transform is $Y(f) = F(f)\hat{W}(f)$, where $\hat{W}(f)$ is the Fourier transform of the output of the impulse generator, defined as:

$$\hat{w}(t) = \sum_{k=-\infty}^{\infty} w_k \delta(t - kT). \tag{S.7}$$

Taking the Fourier transform of the above equation yields:

$$\hat{W}(f) = \sum_{k=-\infty}^{\infty} w_k e^{-j2\pi fkT}. \tag{S.8}$$

We recognize this as the discrete-time Fourier transform $W(e^{j2\pi fT})$ of the sequence w_k. Therefore, we conclude that:

$$\begin{aligned} Y(f) &= F(f)\hat{W}(f) \\ &= F(f)W(e^{j2\pi fT}) \\ &= F(f)H(e^{j2\pi fT})\frac{1}{T}\sum_{m=-\infty}^{\infty} G(f-m/T)X(f-m/T). \end{aligned} \tag{S.9}$$

(b) Yes, you can see from part (a) that if we add two input signals, the output will be a similar superposition.

(c) If $F(f) = 0$ for $|f| > 1/(2T)$ then for $|f| \le 1/(2T)$ we have

$$Y(f)=\frac{1}{T}F(f)H(e^{j2\pi fT})G(f)X(f) \tag{S.10}$$

and the system is time-invariant with frequency response $\frac{1}{T}F(f)H(e^{j2\pi fT})G(f)$.

Problem 2-4. In continuous-time:

$$\begin{aligned} \int_{-\infty}^{\infty} |x(t)|^2 dt &= F.T.\{x(t)x^*(t)\}_{f=0} \\ &= \{X(f) * X^*(-f)\}_{f=0} = \int_{-\infty}^{\infty} |X(f)|^2 df. \end{aligned} \tag{S.11}$$

Discrete-time follows similarly.

Problem 2-5. In continuous-time:

$$\begin{aligned} \int_{-\infty}^{\infty} x(t)y^*(t)dt &= F.T.\{x(t)y^*(t)\}_{f=0} \\ &= \{X(f) * Y^*(-f)\}_{f=0} = \int_{-\infty}^{\infty} X(f)Y^*(f)df. \end{aligned} \tag{S.12}$$

Discrete-time follows similarly.

Problem 2-6. Let $x(t)$ be a continuous-time signal, and let $x_k = x(kT)$ denote the discrete-time signal obtained by sampling at the rate $1/T$. Let E_c and E_d denote the energies of the continuous-time and discrete-time signals, respectively:

$$E_c = \int_{-\infty}^{\infty} x^2(t)dt, \qquad E_d = \sum_k |x_k|^2. \tag{S.13}$$

In the special case when $x(t)$ is properly bandlimited, so that its Fourier transform satisfies $X(f) = 0$ for $|f| > 1/(2T)$, then there is a simple relationship between the two energies:

$$E_d = \sum_k |x_k|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\theta})|^2 d\theta \quad \text{(from Parseval's)}$$

$$= T\int_{-(2T)^{-1}}^{(2T)^{-1}} |X(e^{j2\pi fT})|^2 df \qquad \text{(where } \theta = 2\pi fT\text{)}$$

$$= \frac{1}{T}\int_{-(2T)^{-1}}^{(2T)^{-1}} |\Sigma_m X(f - m/T)|^2 df \qquad \text{(from (2.17))}$$

$$= \frac{1}{T}\int_{-(2T)^{-1}}^{(2T)^{-1}} \Sigma_m |X(f - m/T)|^2 df \quad \text{(from bandlimited assumption)}$$

$$= \frac{1}{T}\Sigma_m \int_{-(2T)^{-1}}^{(2T)^{-1}} |X(f - m/T)|^2 df$$

$$= \frac{1}{T}\Sigma_m \int_{(m-0.5)/T}^{(m+0.5)/T} |X(\nu)|^2 d\nu \qquad \text{(where } \nu = f - m/T\text{)}$$

$$= \frac{1}{T}\int_{-\infty}^{\infty} |X(\nu)|^2 d\nu$$

$$= \frac{E_c}{T}. \tag{S.14}$$

Thus, whenever the signal is properly bandlimited, the continuous-time energy and the discrete-time energy are related by the simple relationship $E_c = E_d T$.

In the general case when the signal is not bandlimited, however, there is no such simple relationship. All three of the following are possible:

(1) $E_c = E_d T$; (2) $E_c < E_d T$; (3) $E_c > E_d T$.

For example, consider a continuous-time signal of the form:

$$x(t) = \Sigma_k x_k g(t - kT) + \Sigma_k y_k g(t - kT - T/2), \tag{S.15}$$

where $g(t)$ is a symmetric unit-height rectangular pulse of width $\tau \le T$, so that $g(t) = 1$ for $|t| < \tau/2$ and $g(t) = 0$ for $|t| > \tau/2$. Note that the energy of $g(t)$ is τ. Because $g(t)$ is time limited, it follows that $x(t)$ is not bandlimited. Sampling $x(t)$ at rate $1/T$ yields $x(kT) = x_k$. For any particular sequence x_k we can make the energy of $x(t)$ big or small by carefully choosing the pulse width τ and the secondary sequence y_k. In particular, the energy of $x(t)$ is easily expressed in terms of $E_d = \Sigma_k |x_k|^2$ and $E_y = \Sigma_k |y_k|^2$, namely:

$$E_c = \int_{-\infty}^{\infty} x^2(t) dt$$

$$= \int_{-\infty}^{\infty} (\Sigma_k x_k g(t - kT) + \Sigma_k y_k g(t - kT - T/2))^2 dt$$

$$= \tau(\Sigma_k |x_k|^2 + \Sigma_k |y_k|^2)$$

$$= \tau(E_d + E_y). \tag{S.16}$$

$\Rightarrow$ (1) We can get $E_c = E_d T$ by choosing $\tau = T$ and $y_k = 0$.

(2) We can get $E_c < E_d T$ by choosing $\tau = T/2$ and $y_k = 0$.

(3) We can get $E_c > E_d T$ by choosing $\tau = T/2$ and $y_k = 2x_k$.

Problem 2-7. The transfer function is $H(z) = 1 + z^{-1}$. Therefore, the frequency response is:

$$H(e^{j\theta}) = 1 + e^{-j\theta} = e^{-j\theta/2}(e^{j\theta/2} + e^{-j\theta/2})$$
$$= 2e^{-j\theta/2}\cos(\theta/2) \ . \tag{S.17}$$

(a) When θ is restricted to $|\theta| \le \pi$, the magnitude response and phase response are:

$$|H(e^{j\theta})| = 2\cos(\theta/2) \ ,$$
$$\angle H(e^{j\theta}) = -\theta/2. \tag{S.18}$$

These repeat with period 2π for other θ. Clearly, the phase response is a linear function of θ over this range, and it is thus a piecewise linear function for all θ.

(b) Assume $|\theta_0| \le \pi$, without loss of generality. (If not, simply add or subtract multiples of 2π until it is; this process won't change the input sequence h_k.) When the input is $h_k = \cos(\theta_0 k) = 0.5e^{j\theta_0 k} + 0.5e^{-j\theta_0 k}$, a sum of eigenfuctions, the output is:

$$y_k = 0.5H(e^{j\theta_0})e^{j\theta_0 k} + 0.5H(e^{-j\theta_0})e^{-j\theta_0 k}$$
$$= \cos(\theta_0/2)e^{j(\theta_0 k - \theta_0/2)} + \cos(\theta_0/2)e^{-j(\theta_0 k - \theta_0/2)}$$
$$= 2\cos(\theta_0/2)\cos(\theta_0 k - \theta_0/2) \ . \tag{S.19}$$

The output is thus a sinusoid whose magnitude and phase are given by the magnitude response and phase response, respectively, evaluated at the sinusoid's frequency θ_0.

Problem 2-8. The Fourier transform of a real system is conjugate symmetric, so

$$H(f) = A(f)e^{j\theta(f)} = H^*(-f) = A(f)e^{-j\theta(-f)}, \tag{S.20}$$

since $A(f) = |H(f)|$ is both non-negative and symmetric. Hence, $\theta(f) = -\theta(-f)$.

Problem 2-9. From Problem 2-8, the phase response of a real system is anti-symmetric, so the frequency response of the phase shifter should be

$$H(f) = e^{-j\phi \operatorname{sgn}(f)} = \cos(\phi) + j\sin(\phi)\operatorname{sgn}(f), \tag{S.21}$$

where we have used the symmetry and anti-symmetry of the cos and sin, respectively. Thus, in the time domain this becomes

$$h(t) = \delta(t)\cos(\phi) - \frac{1}{\pi t}\sin(\phi). \tag{S.22}$$

Problem 2-10. If $\mathrm{Re}\{a\} > 0$, then we can use the Fourier transform pair

$$y(t) = \frac{1}{jt + a} \quad \leftrightarrow \quad Y(f) = 2\pi e^{2\pi a f}u(-f), \tag{S.23}$$

where $u(f)$ is the unit step function. Then we observe that $x(t)$ is $y(t)$ convolved with an impulse stream $\sum_{m=-\infty}^{\infty} \delta(t - mT)$, so its transform is

$$X(f) = Y(f)\frac{1}{T}\sum_{m=-\infty}^{\infty} \delta(f - m/T)$$

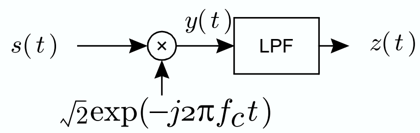
$$= \frac{2\pi}{T}\sum_{m=-\infty}^{\infty} \delta(f - m/T)e^{2\pi am/T}. \tag{S.24}$$

If $\mathrm{Re}\{a\} = 0$, then we can use the transform of $1/jt$, namely $-\pi\,\mathrm{sign}(f)$, convolving it again with an impulse stream to get

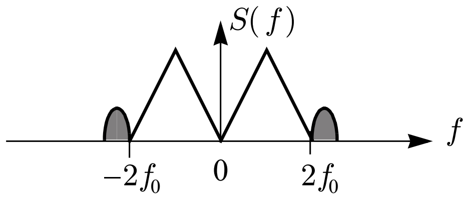
$$X(f) = \frac{-\pi}{T}\,\mathrm{sign}(f)\sum_{m=-\infty}^{\infty} \delta(f - m/T), \qquad (\text{when } \mathrm{Re}\{a\} = 0). \tag{S.25}$$

Problem 2-11. Given $Y(f) = H(f)X(f)$, then $X(f_0) = 0$ necessarily implies that $Y(f_0) = 0$.

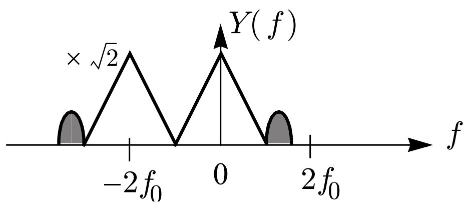
Problem 2-12. It is easy to see that the two outputs of Fig. 2-6(a) are the real and imaginary parts of a complex signal $z(t)$, where $z(t)$ is the output of the system sketched below:



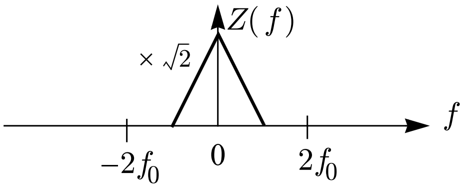
Schematically let us represent the input Fourier transform $S(f)$ as follows:



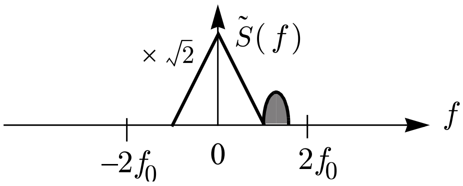
The triangles represents the portion of $S(f)$ bandlimited to $|f| \le 2f_0$, and the shaded lumps represent any energy beyond $2f_0$. According to the above system diagram, the input to the LPF has Fourier transform:



And thus the LPF output will have Fourier transform:



Contrast this with the Fourier transform of the complex envelope of $s(t)$, namely:



Comparing $Z(f)$ to $\tilde{S}(f)$, we see they will be equal if and only if the shaded lump is zero, i.e., if and only if there is no energy beyond $2f_0$.

Problem 2-18.

(a) From (2.7),

$$y_k = \sum_{m=-\infty}^{\infty} z^{k-m} h_m = z^k \sum_{m=-\infty}^{\infty} z^{-m} h_m = z^k H(z). \tag{S.26}$$

Because the system is time invariant, $H(z)$ does not depend on k.

(b) If we constrain z to lie on the unit circle, $|z| = 1$, the two results are identical.

Problem 2-19. Let the response to z^k be y_k. By linearity, if the input to the system is z^{k+m} then the output is $z^m y_k = z^k z^m$. By time invariance the response to that same input is y_{k+m}. Setting these two responses equal,

$$z^m y_k = y_{k+m} \tag{S.27}$$

and setting $k = 0$ we get the desired result

$$y_m = y_0 z^m. \tag{S.28}$$

The transfer function is a complex number y_0, which is evidently a function of z, so we define the notation $y_0 = H(z)$ to reflect this property.

Problem 2-20. The Z transform is

$$X(z) = \sum_{k=-\infty}^{\infty} z^{-k} a^k u_k = \sum_{k=0}^{\infty} (az^{-1})^k. \tag{S.29}$$

For any b satisfying $|b| < 1$ we have the identity

$$\frac{1}{1-b} = \sum_{k=0}^{\infty} b^k \tag{S.30}$$

which is easily verified by using long division on the left hand side. Therefore, in the region of the complex plane where $|z^{-1}| < 1/|a|$, the Z transform is

$$X(z) = \frac{1}{1 - az^{-1}}. \tag{S.31}$$

Outside of this region, the Z transform does not exist. If $|a| > 1$, we easily see that the signal goes to infinity as k increases. Not coincidentally, if $|a| > 1$ the region in which the Z transform exists does not include the unit circle, implying that the Fourier transform does not exist either.

Problem 2-21. If the response of the system to z^t is $y(t)$, then by linearity the response to $z^{t+u} = z^t z^u$ is $z^u y(t)$. By time invariance, the response to z^{t+u} is $y(t = u)$. Setting these two equal, $y(t + u) = t^u y(t)$, and setting $t = 0$, $y(u) = y(0) z^u$.

Clearly $e^{st} = z^t$ if $z = e^s$, i.e., if $s = j2\pi f$, then $z = e^{j2\pi f}$, a point on the unit circle.

Substituting into the convolution,

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(\tau) x(t - \tau) d\tau \\ &= \int_{-\infty}^{\infty} h(\tau) e^{s(t-\tau)} d\tau \\ &= e^{st} \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau = e^{st} H_L(s) \end{aligned} \tag{S.32}$$

which implies that $H_L(s)$ is an eigenvalue of the system.

Problem 2-22.

(a) In both cases, we have:

$$X(z) = \frac{z}{z - a}. \tag{S.33}$$

(b) $|z| > |a|$ and $|z| < |a|$ respectively.

(c) $|a| < 1$ and $|a| > 1$ respectively.

Problem 2-23. When the ROC is $|z| > |a|$, which implies $|az^{-1}| < 1$, we get:

$$\frac{1}{1 - az^{-1}} = 1 + az^{-1} + a^2 z^{-2} + \ldots, \tag{S.34}$$

so that the time domain signal can be read off from the polynomial coefficients as $a^k u_k$.
On the other hand, when the ROC is $|z| < |a|$, which implies $|za^{-1}| < 1$, we get:

$$\frac{-z/a}{1 - z/a} = \frac{-z}{a}\left(1 + \frac{z}{a} + \left(\frac{z}{a}\right)^2 + \ldots\right) \tag{S.35}$$

In this case, the time-domain coefficients are $-a^k u_{-k-1}$.

Problem 2-24. First we perform a partial fraction expansion,

$$X(z) = \frac{A}{z-a} + \frac{B}{z-b} \tag{S.36}$$

where

$$A = \frac{a^2}{a-b}, \quad B = \frac{b^2}{b-a}. \tag{S.37}$$

(a) Causality implies that the ROC is $|z| > |b|$. Taking the inverse transform of each term in the partial fraction expansion and then adding the results yields,

$$x_k = Aa^{k-1} + Bb^{k-1} \tag{S.38}$$

for $k \geq 1$, and zero otherwise.

(b) If the signal is two-sided, the ROC must be be $|a| < |z| < |b|$ and hence:

$$x_k = \begin{cases} Aa^k, & \text{for } k \geq 0 \\ -Bb^k, & \text{for } k < 0 \end{cases}. \tag{S.39}$$

(c) For (a) the signal is not stable because $b^k \to \infty$. This is because the ROC does not include the unit circle. For (b) the ROC does include the unit circle so the signal is stable (this is the only ROC for which the signal is stable).

Problem 2-25. Consider the zeros, and the poles will follow similarly. Assume that z_0 is a zero, and hence

$$\sum_{k=0}^{M} b_k z_0^{-k} = 0. \tag{S.40}$$

Ttaking the conjugate of this equation:

$$\sum_{k=0}^{M} b_k^*(z_0^*)^{-k} = \sum_{k=0}^{M} b_k(z_0^*)^{-k} = 0 \tag{S.41}$$

and we conclude that z_0^* is also a zero.

Problem 2-26. The easy terms are

$$H_{\text{zero}}(z) = (1 - jz^{-1})(1 + jz^{-1}) = (1 + z^{-2}), \tag{S.42}$$

$$H_{\min}(z) = \left(\frac{1}{1-0.5e^{j\pi/8}z^{-1}}\right)\left(\frac{1}{1-0.5e^{-j\pi/8}z^{-1}}\right) = \frac{1}{1-\cos(\pi/8)z^{-1}+0.25z^{-2}}, \tag{S.43}$$

but the maximum-phase term requires some more work. Writing one maximum-phase zero in monic form,

$$(1 - \tfrac{3}{2}e^{j\pi/8}z^{-1}) = z^{-1}(\tfrac{-3}{2}e^{j\pi/8})(1 - \tfrac{2}{3}e^{-j\pi/8}z) \tag{S.44}$$

and considering both zeros we get $L = -2$, $B = (-2/3)^2 = 4/9$, and

$$H_{\max}(z) = 1 - \tfrac{4}{3}\cos(\pi/8)z + \tfrac{4}{9}z^2. \tag{S.45}$$

Problem 2-27. Hypothesize a zero-based spectrum $S(z) = (1 - bz^{-1})(1 - bz)$, where b is real and $-1 < b < 1$; this is already in the form of the spectral factorization (2.67). Therefore, its geoemetric mean is unity by inspection. From Parseval's, its arithmetic mean is:

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi} S(e^{j\theta})\,d\theta &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |1 - be^{-j\theta}|^2 d\theta \\ &= \sum_k |\delta_k - b\delta_{k-1}|^2 = 1 + b^2.\end{aligned} \tag{S.46}$$

There is no way this will be three, given our constraint that $|b| < 1$. So this was a dead end. Try instead a pole-based spectrum:

$$S(z) = \frac{1}{(1 - az^{-1})(1 - az)}, \tag{S.47}$$

where a is real and $|a| < 1$; this too is in the form of the spectral factorization (2.67), and its geometric mean is clearly unity. Its arithmetic mean is:

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi} S(e^{j\theta})\,d\theta &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\frac{1}{1 - ae^{-j\theta}}\right|^2 d\theta \\ &= \sum_k |a^k u_k|^2 = \frac{1}{1 - a^2}.\end{aligned} \tag{S.48}$$

Thus, we can make the arithmetic mean of (S.47) equal to 3 by choosing $a = \sqrt{2/3}$.

Problem 2-28.

(a) Let

$$A(z) = \frac{z^{-1} - c^*}{1 - cz^{-1}} \tag{S.49}$$

and note that $G(z) = H(z)A(z)$. Then $A(z)$ is allpass (Example 2-9), so

$$|G(e^{j\theta})| = |H(e^{j\theta})|\cdot|A(e^{j\theta})| = |H(e^{j\theta})|. \tag{S.50}$$

It has the effect of moving a zero $|c| < 1$ outside the unit circle without changing the magnitude response.

(b) The hint tells us that h_k can be obtained by passing f_k through a filter with transfer function $(1 - cz^{-1})$, so that

$$h_k = f_k - cf_{k-1}. \tag{S.51}$$

Similarly g_k can be obtained by putting f_k through transfer function $(z^{-1} - c^*)$,

$$g_k = f_{k-1} - c^* f_k. \tag{S.52}$$

Calculating the difference between the energy in each sample in h_k and g_k,

$$|h_k|^2 - |g_k|^2 = |f_k - cf_{k-1}|^2 - |f_{k-1} - c^* f_k|^2 = (1 - |c|^2)(|f_k|^2 - |f_{k-1}|^2) \tag{S.53}$$

and calculating the difference in energy for the first N samples,

$$\sum_{k=0}^{N} (|h_k|^2 - |g_k|^2) = (1 - |c|^2)|f_N|^2 \ge 0, \tag{S.54}$$

which exploits the fact that f_k is causal (why?), and $|c| < 1$, and hence $f_{-1} = 0$.

Problem 2-29. Using the allpass transfer function in Example 2-9 with $|c| < 1$, define a signal w_k with Z transform $X(z)/(1 - cz^{-1})$. Note that w_k is causal also (why?). Then x_k is obtained by putting w_k through a system with transfer function $(1 - cz^{-1})$ and y_k is obtained by system $(z^{-1} - c^*)$. From here, the derivation is identical to Problem 2-28, with the result that:

$$\sum_{k=0}^{N} (|x_k|^2 - |y_k|^2) = (1 - |c|^2)|w_N|^2 \ge 0 \ . \tag{S.55}$$

Problem 2-30. Using (2.52) to factor $H(z)$ into the product of minimum-phase and maximum-phase transfer functions,

$$\begin{aligned} H(z) &= B \cdot z^L \cdot H_{\text{min}}(z) \cdot H_{\text{max}}(z) \cdot H_{\text{zero}}(z) \\ &= B \cdot z^L \cdot H_{\text{min}}(z) H_{\text{max}}^*(1/z^*) \frac{H_{\text{max}}(z)}{H_{\text{max}}^*(1/z^*)} \ . \end{aligned} \tag{S.56}$$

Now note that $H_{\text{max}}^*(1/z^*)$ is minimum phase. Furthermore, from Example 2-10, the ratio $H_{\text{max}}(z)/H_{\text{max}}^*(1/z^*)$ is allpass, as is z^L. Hence,

$$A(z) = z^L \frac{H_{\text{max}}(z)}{H_{\text{max}}^*(1/z^*)} \tag{S.57}$$

and

$$H'_{\text{min}}(z) = B H_{\text{min}}(z) H_{\text{max}}^*(1/z^*) \ . \tag{S.58}$$

Problem 2-31. From the Fourier transform tables, $g(t) = h^*(-t) \leftrightarrow G(f) = H^*(f)$.

Problem 2-32.

(a) The squared norms (i.e., energies) of both signals is unity, $E_1 = E_2 = 1$. The inner product is

$$\int_{-\infty}^{\infty} s_1(t) s_2(t) dt = 1 \cdot \frac{3}{4} - 1 \cdot \frac{1}{4} = \frac{1}{2} \ . \tag{S.59}$$

From the discussion on page 43, the angle θ between $s_1(t)$ and $s_2(t)$ satisfies:

$$\sqrt{E_1}\,\sqrt{E_2}\cos\theta = \int_{-\infty}^{\infty} s_1(t)s_2(t)\,dt. \tag{S.60}$$

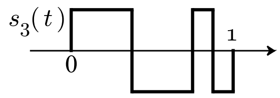
Since the energies are both unity, (S.59) implies that $\cos(\theta) = 1/2$, or $\theta = \pi/3 = 60°$.

(b) The squared norm of the sum is:

$$\int_{-\infty}^{\infty} (s_1(t) + s_2(t))^2 dt = E_1 + E_2 + 2\int_{-\infty}^{\infty} s_1(t)s_2(t)\,dt$$
$$= 1 + 1 + 2\tfrac{1}{2} = 3. \tag{S.61}$$

Hence, the norm is $\sqrt{3}$.

(c) There are many possibilities, but here is one:



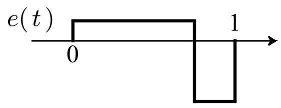
(d) Since $s_1(t)$ is unit energy, it is a basis function for the subspace it spans. Consider $\hat{s}_2(t)$, the projection of $s_2(t)$ onto the subspace spanned by $s_1(t)$; is given by $s_{2,1}s_1(t)$, where from (S.59) the coefficient is:

$$s_{2,1} = \int_{-\infty}^{\infty} s_1(t)s_2(t)\,dt = \frac{1}{2}. \tag{S.62}$$

Therefore, the projection error is given by:

$$e(t) = s_2(t) - \hat{s}_2(t)$$
$$= s_2(t) - \tfrac{1}{2}s_1(t), \tag{S.63}$$

as sketched below:



This signal is 0.5 for $t \in [0, 3/4)$ and –1.5 for $t \in [3/4, 1)$. Being a linear combination of $s_1(t)$ and $s_2(t)$, $e(t)$ is clearly in their span. And from the projection theorem (Theorem 2-2), the signal $e(t)$ is orthogonal to $s_1(t)$. So the answer is $s_4(t) = e(t)$, as sketched above.

(e) The answer is $\hat{s}_5(t)$, the projection of $s_5(t)$ onto the subspace spanned by $s_1(t)$ and $s_2(t)$. It remains only to find this projection. Normalizing $e(t)$ yields the pair of basis functions $\phi_1(t) = s_1(t)$, $\phi_2(t) = \frac{2}{\sqrt{3}}e(t)$ for the subspace spanned by $s_1(t)$ and $s_2(t)$. The projection of $s_5(t)$ onto the two basis signals is

$$\int_{-\infty}^{\infty} s_5(t)\phi_1(t)\,dt = -\frac{1}{2}, \qquad \int_{-\infty}^{\infty} s_5(t)\phi_2(t)\,dt = \frac{1}{2\sqrt{3}}, \tag{S.64}$$

and hence the projection on the subspace is

$$\hat{s}_5(t) = -\frac{1}{2}\phi_1(t) + \frac{1}{2\sqrt{3}}\phi_2(t) = -\frac{1}{2}s_1(t) + \frac{1}{3}e(t). \tag{S.65}$$

Problem 2-33.

(a) Clearly if two signals are bandlimited, then their weighted sum is also bandlimited. Being closed and a subset, it is itself a subspace.

(b) Let $x(t)$ be in the subspace. By Parseval's theorem (Problem 2-4), for any $y(t) \in B$,

$$\int_{-\infty}^{\infty} x(t)y^*(t)dt = \int_{-W}^{W} X(f)Y^*(f)df = 0. \tag{S.66}$$

Clearly, this is satisfied if and only if $X(f) = 0$ for $|f| \le W$.

(c) Let this projection be $\hat{s}_1(t)$, then $\langle s_1(t) - \hat{s}_1(t), y(t)\rangle = 0$ for all $y(t) \in B$. From (b) this implies that $S_1(f) = \hat{S}_1(f)$ for $|f| \le W$, and of course since $\hat{s}_1(t) \in B$, we must have that $P(f) = 0$ for $|f| > W$. Hence,

$$S_1(f) = \int_0^1 e^{-j2\pi ft}dt = \left(\frac{1 - e^{-j2\pi f}}{j2\pi f}\right) \tag{S.67}$$

and

$$\hat{s}_1(t) = \int_{-1}^{1} \left(\frac{1 - e^{-j2\pi f}}{j2\pi f}\right) e^{j2\pi ft} df. \tag{S.68}$$

Unfortunately this integral cannot be evaluated in closed form.

Problem 2-34. Let $X_1 \in \mathcal{S}_1$ and $X_2 \in \mathcal{S}_2$. An element of $\mathcal{S}_1 \oplus \mathcal{S}_2$ can be written in the form $X_1 + X_2$. Hence it suffices to show that

$$\langle X - (P_{\mathcal{S}_1}(X) + P_{\mathcal{S}_2}(X)), X_1 + X_2\rangle = 0. \tag{S.69}$$

Expanding the left side, it equals

$$\langle X - P_{\mathcal{S}_1}(X), X_1\rangle - \langle P_{\mathcal{S}_2}(X), X_1\rangle + \langle X - P_{\mathcal{S}_2}(X), X_2\rangle - \langle P_{\mathcal{S}_1}(X), X_2\rangle. \tag{S.70}$$

But by the definition of projection

$$\langle X - P_{\mathcal{S}_1}(X), X_1\rangle = \langle X - P_{\mathcal{S}_2}(X), X_2\rangle = 0, \tag{S.71}$$

and because $\mathcal{S}_1$ and $\mathcal{S}_2$ are orthogonal subspaces

$$\langle P_{\mathcal{S}_2}(X), X_1\rangle = \langle P_{\mathcal{S}_1}(X), X_2\rangle = 0. \tag{S.72}$$

The result is thus established.

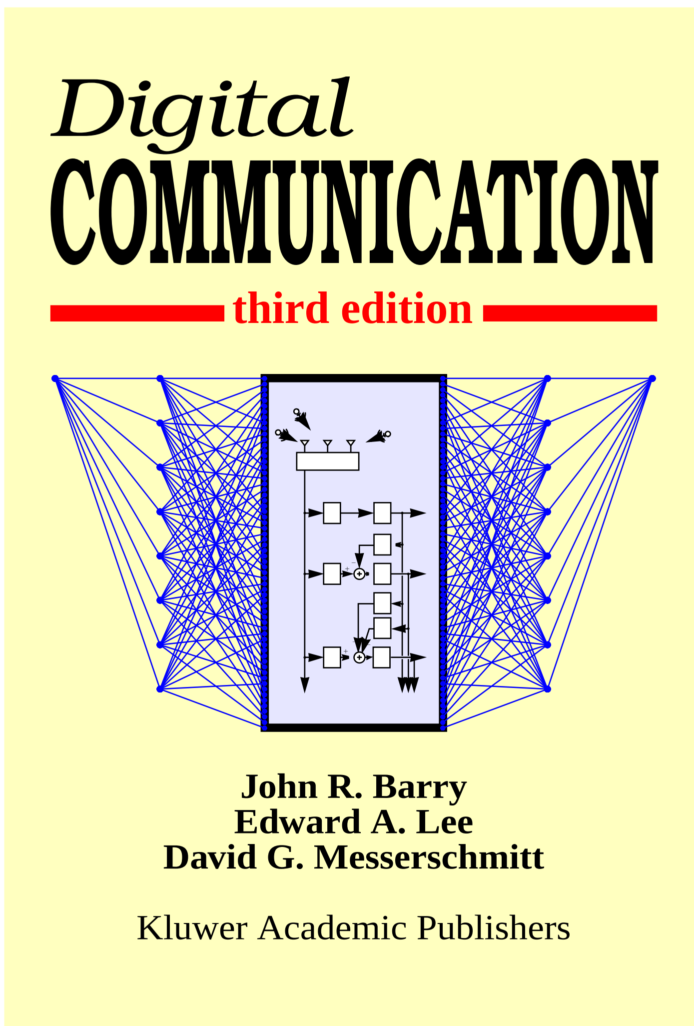
Problem 2-35. The Schwarz inequality states that

$$|\rho_h(k)|^2 \le \left(\int_{-\infty}^{\infty} |h(t)|^2 dt \right) \left(\int_{-\infty}^{\infty} |h(t - kT)|^2 dt \right), \tag{S.73}$$

and since it is easy to verify that the signal and its time-translate have identical norms, this becomes the desired result.

Solutions Manual

for



Chapter 3

Problem 3-1. Taking the first derivative,

$$\frac{\partial}{\partial s}\Phi_X(s) = \Phi_X(s)(\mu + \sigma^2 s) \tag{S.71}$$

and setting $s = 0$ we get $E[X] = \mu$. Similarly, the second derivative is

$$\frac{\partial^2}{\partial s^2}\Phi_X(s) = \frac{\partial}{\partial s}\Phi_X(s)(\mu + \sigma^2 s) + \Phi_X(s)\sigma^2 , \tag{S.72}$$

and again setting $s = 0$ we get $E[X^2] = (\mu^2 + \sigma^2)$. The variance is therefore σ^2.

Problem 3-2. We can use (3.17) and carry out the integral.

Problem 3-3.

$$\begin{aligned}\sqrt{2\pi}\, Q(x) &= \int_x^\infty \frac{1}{t} te^{-t^2/2}\, dt = \frac{-1}{t} e^{-t^2/2} \Big|_x^\infty - \int_x^\infty \frac{1}{t^2} e^{-t^2/2}\, dt \\ &= \frac{1}{x} e^{-x^2/2} - \int_x^\infty \frac{1}{t^2} e^{-t^2/2}\, dt \quad .\end{aligned} \tag{S.73}$$

The bounds then follow from the fact that

$$0 < \int_x^\infty \frac{1}{t^2} e^{-t^2/2}\, dt \le \frac{1}{x^3}\int_x^\infty te^{-t^2/2}\, dt = \frac{1}{x^3} e^{-x^2/2} \quad . \tag{S.74}$$

Problem 3-4.

(a) Define the MSE as ε^2, so that:

$$\varepsilon^2 = E[\,|X - aY|^2] = E[\,|X|^2] + |a|^2 E[\,|Y|^2] - 2\mathrm{Re}\{a^* E[XY^*]\}. \tag{S.75}$$

Setting to zero the derivative with respect to the real and imaginary parts, yields:

$$\mathrm{Re}\{a\}E[\,|Y|^2] = \mathrm{Re}\{E[XY^*]\}, \tag{S.76}$$

$$\mathrm{Im}\{a\}E[\,|Y|^2] = \mathrm{Im}\{E[XY^*]\}. \tag{S.77}$$

Combining these two equations yields:

$$a = E[XY^*] / E[\,|Y|^2] \quad . \tag{S.78}$$

(b) The restated problem: Find the RV in the subspace spanned by Y that is closest to X.

(c) By the orthogonality principle,

$$\langle X - aY,\ Y\rangle = 0 \tag{S.79}$$

or

$$a = \langle X,Y\rangle / \langle Y,Y\rangle = E[XY^*] / E[\,|Y|^2] \quad . \tag{S.80}$$

Problem 3-5.

(a) First, we can write:

$$\begin{aligned} \mathrm{E}[\,|E'_k\,|]^2 &= \mathrm{E}[\,|E'_k - E_k + E_k\,|^2] \\ &= \mathrm{E}[\,|E'_k - E_k\,|^2] + \mathrm{E}[\,|E_k\,|^2] + 2\mathrm{Re}\{(E'_k - E_k)E_k{}^*\}. \end{aligned} \tag{S.81}$$

Since the filters generating both E_k and E'_k have unity coefficients at delay zero, the filter generating $(E'_k - E_k)$ has a zero coefficient at delay zero, and this signal is a function of only *past* inputs $X_{k-1}, X_{k-2}, \ldots$, and in view of property (3.204), the third term in (S.416) must in fact be zero.

(b) Since E_k is a linear combination of $X_k, X_{k-1}, \ldots$, it follows from (3.204) that

$$E[E_{k+m}E_k{}^*] = R_E(m) = 0 \;, \qquad m > 0, \tag{S.82}$$

and since the autocorrelation function has conjugate symmetry, it follows that $R_E(m) = 0$ for all $m \neq 0$.

Problem 3-6.

(a) The problem can be restated as:
Given a sequence of vectors $\boldsymbol{X}_k$, $-\infty < k < \infty$, with inner products

$$R_X(m) = \langle \boldsymbol{X}_{k+m}, \boldsymbol{X}_k \rangle \tag{S.83}$$

that are independent of k, given a subspace $\mathcal{M}$ spanned by $\{\boldsymbol{X}_{k-m},\ m > 0\}$, find the vector $\ddot{X}_{\boldsymbol{k}}$ in $\mathcal{M}$ that is closest to the vector $\boldsymbol{X}_k$, with error vector $\boldsymbol{E}_k = \boldsymbol{X}_k - \hat{\boldsymbol{X}}_{\boldsymbol{k}}$.

(b) By the projection theorem, for every vector $\boldsymbol{Y} \in \mathcal{M}$, we must have that

$$\langle \boldsymbol{E}_k, \boldsymbol{Y} \rangle = 0 \tag{S.84}$$

and in particular

$$\langle \boldsymbol{E}_k, \boldsymbol{X}_{k-m} \rangle = 0,\ \ m > 0 \;. \tag{S.85}$$

Thus the prediction error vector is orthogonal to the past (the vectors used in the prediction estimation). Since $E_{k-m} \in \mathcal{M}$, $m > 0$, it follows that

$$\langle \boldsymbol{E}_k, \boldsymbol{E}_{k-m} \rangle = 0,\ \ m > 0 \;, \tag{S.86}$$

or equivalently $R_E(m) = 0$, $m \neq 0$.

Problem 3-7.

$$R_Y(0) = E[\,|\,Y_k\,|^2] = E[\,|X(kT)\,|^2] \;. \tag{S.87}$$

Observe that since $X(t)$ is WSS,

$$R_X(0) = E[\,|X(t)\,|^2] = E[\,|X(t+\tau)\,|^2] \tag{S.88}$$

for any τ. Define

$$\tau = kT - t\,, \tag{S.89}$$

and the result follows.

Problem 3-8. Mechanically,

$$E[A_p A_q A_r A_s] = \delta_{pq}\delta_{rs} + \delta_{pr}\delta_{qs} + \delta_{ps}\delta_{qr} - 2\delta_{pq}\delta_{pr}\delta_{ps}\,. \tag{S.90}$$

Problem 3-9. Observe from (3.67) that

$$\int_{f_a}^{f_b} S_X(f)\,df = \int_{-\infty}^{\infty} S_Y(f)\,df \tag{S.91}$$

and from the definition of the power spectrum,

$$R_Y(\tau) = \int_{-\infty}^{\infty} S_Y(f)e^{j2\pi f\tau}\,df, \tag{S.92}$$

from which the first result follows. To show that $S_X(f)$ is non-negative everywhere, assume it is negative over some region from f_a to f_b. Then note that

$$\int_{f_a}^{f_b} S_X(f)df \tag{S.93}$$

must also be negative, which implies that $R_Y(0)$ is negative. But

$$R_Y(0) = \mathrm{E}[\,|\,Y(t)\,|^2]\,, \tag{S.94}$$

which must be non-negative, a contradiction.

Problem 3-10. The power spectrum is

$$N_0 \cdot |F(e^{j\theta})|^2 = N_0 \cdot F(e^{j\theta})F^*(e^{j\theta}) \tag{S.95}$$

where $F(e^{j\theta})$ is the discrete-time Fourier transform of $f(kT)$. Since the inverse Fourier transform of $F^*(e^{j\theta})$ is $f^*(-kT)$, the result follows immediately.

Problem 3-11.

$$R_{XY}(\tau) = E[X(t+\tau)Y^*(t)] = E[Y^*(p-\tau)X(p)] = R_{YX}{}^*(-\tau). \tag{S.96}$$

No, it may be complex-valued.

Problem 3-12. If the steady state probabilities exist they must satisfy (3.117), and also must satisfy:

$$\textstyle\sum_{j \in \Omega} p(j) = 1\ . \tag{S.97}$$

Define the row vector of state probabilities

$$\boldsymbol{p}_k = [p_k(0), \ldots,\ p_k(M)]\quad . \tag{S.98}$$

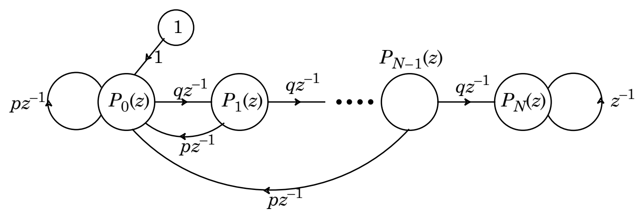
Then the system of equations in (3.117) can be written more concisely as

$$\boldsymbol{p}_{k+1} = \boldsymbol{p}_k \mathbf{P} . \tag{S.99}$$

The condition that the state probabilities don't change through a state transition gives the desired steady-state probabilities.

Problem 3-13.

(a) The signal flow graph is shown below:



(b) The corresponding equations are

$$P_0(z) = 1 + \sum_{i=0}^{N-1} pz^{-1} P_i(z) \tag{S.100}$$

$$P_i(z) = qz^{-1} P_{i-1}(z) , \quad 1 \le i \le N-1 \tag{S.101}$$

$$P_N(z) = qz^{-1} P_{N-1}(z) + z^{-1} P_N(z) \ . \tag{S.102}$$

Solving these equations and using the identity

$$\sum_{i=0}^{N-1} r^i = \frac{1-r^N}{1-r} , \tag{S.103}$$

we get the result.

(c) This follows directly by differentiation.

(d) For this case,

$$f_N \approx q^{-N} \tag{S.104}$$

which is what we would expect. The probability of a head is q, the probability of N heads in a row is q^N, and on a relative frequency basis N heads in a row will occur once out of q^{-N} trials.

Problem 3-14.

$$p(\psi_0, \psi_1, \dots, \psi_n) = p(\psi_n \mid \psi_{n-1}, \dots, \psi_0) p(\psi_{n-1}, \dots, \psi_0) \tag{S.105}$$

$$= p(\psi_n \mid \psi_{n-1}) p(\psi_{n-1}, \dots, \psi_0) \tag{S.106}$$

$$= p(\psi_n \mid \psi_{n-1}) p(\psi_{n-1} \mid \psi_{n-2} \dots, \psi_0)\ p(\psi_{n-2}, \dots, \psi_0) \tag{S.107}$$

$$= p(\psi_n \mid \psi_{n-1}) p(\psi_{n-1} \mid \psi_{n-2}) p(\psi_{n-2}, \dots, \psi_0) \tag{S.108}$$

$$= \dots \tag{S.109}$$

Problem 3-15.

$$p(\psi_n \mid \psi_{n+1},\ldots,\psi_{n+m}) = \frac{p(\psi_n, \psi_{n+1}, \ \ldots \ \psi_{n+m})}{p(\psi_{n+1}, \ldots \psi_{n+m})}. \tag{S.110}$$

Using the result in Problem 3-14 on both the numerator and denominator and canceling the terms that are equal, this becomes

$$\begin{aligned} p(\psi_n \mid \psi_{n+1},\ldots,\psi_{n+m}) &= \frac{p(\psi_{n+1}|\psi_n)p(\psi_n)}{p(\psi_{n+1})} \\ &= p(\psi_n \mid \psi_{n+1}). \end{aligned} \tag{S.111}$$

Problem 3-16.

$$p(\psi_n,\psi_s \mid \psi_r) = \frac{p(\psi_n, \psi_r, \psi_s)}{p(\psi_r)}$$

$$= p(\psi_n \mid \psi_r)\left(\frac{p(\psi_r|\psi_s)p(\psi_s)}{p(\psi_r)}\right),$$

where the last equality follows fromProblem 3-14. The result now follows using Bayes' rule.

Problem 3-17. The signal flow graph is in shown in Fig. S-1(a). The set of equations governing this signal flow graph is

$$P_0(z) = 0.5z^{-1}P_0(z) + 0.5z^{-2}P_1(z) + 1 \tag{S.112}$$

$$P_1(z) = 0.5z^{-1}P_1(z) + 0.5P_0(z) \ . \tag{S.113}$$

Solving these linear equations for $P_0(z)$,

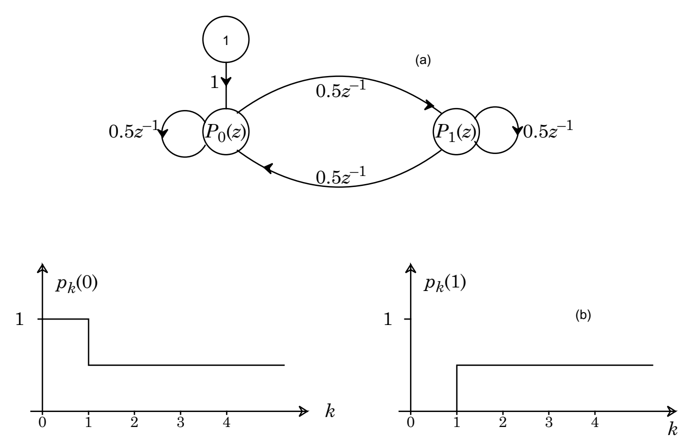


Fig. S-1. (a) The signal flow graph describing the state probabilities of the parity check example when the initial state is zero. (b) The state probabilities as a function of k.

$$P_0(z) = \frac{1 - 0.5z^{-1}}{1 - z^{-1}} = \frac{1}{1 - z^{-1}} - \frac{0.5z^{-1}}{1 - z^{-1}} .$$

Inverting the Z transform,

$$p_k(0) = u_k - 0.5u_{k-1},$$

where u_k is the unit step. This is sketched in Fig. S-1(b). Computing $P_1(z)$ similarly,

$$P_1(z) = \frac{0.5z^{-1}}{1 - z^{-1}} .$$

Inverting the Z transform:

$$p_k(1) = 0.5u_{k-1}.$$

The Markov chain is not stationary.

Problem 3-18. The Markov state diagram is shown in Fig. S-2(a).

(a) The diagram shows the independence required for the random process to be Markov, assuming that the coin tosses are independent of one another.

(b) The signal flow graph is shown in Fig. S-2(b).

(c) Writing the set of equations and solving them,

$$\frac{P_2(z)}{1} + 1 = \frac{z}{4z^3 - 6z^2 + z + 1} . \tag{S.114}$$

Problem 3-19. A Poisson random variable N with parameter a has the pmf given in (3.140). It is easy to show (see Exercise 3-19-(b)) that the moment generating function for N is:

$$\Phi_N(s) = \exp\{a(e^s - 1)\}. \tag{S.115}$$

Therefore, (3.26) indicates that the optimum value of s satisfies:

$$\frac{x}{a} = e^s . \tag{S.116}$$

Plugging this into the bound gives the result.

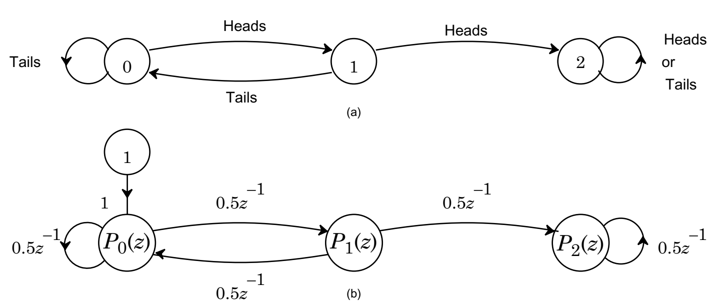


Fig. S-2. (a) Markov chain description of the coin toss experiment.
(b) Signal flow graph.

Problem 3-20. The solution follows immediately from the fact that $N(t_0)$ is Poisson distributed with parameter $a = \lambda t_0$. Hence from (3.141) we get

$$E[N(t_0)] = \lambda t_0 \,, \tag{S.117}$$

$$Var[N(t_0)] = \lambda t_0 \,. \tag{S.118}$$

Problem 3-21. Writing the joint probability as a conditional probability, the desired probability is

$$p_{N(t_2)\,|\,N(t_1)}(k+n\,|\,k)p_{N(t_1)}(k) \quad . \tag{S.119}$$

The left term is the probability of n arrivals in time interval $(t_2 - t_1)$, or

$$\frac{(\lambda(t_2 - t_1))^k}{k!}e^{-\lambda(t_2 - t_1)} \tag{S.120}$$

and the second probability is

$$\frac{(\lambda t_1)^k}{k!}e^{-\lambda t_1} \,. \tag{S.121}$$

Taking the product, the result follows immediately.

Problem 3-22. The process is governed by the differential equation

$$\frac{dq_j(t)}{dt} = (j-1)\lambda q_{j-1}(t) - j\lambda q_j(t) \tag{S.122}$$

which has Laplace transform

$$Q_j(s) = \frac{q_j(0) + (j-1)\lambda Q_{j-1}(s)}{s + j\lambda} \tag{S.123}$$

and iterating we get

$$Q_j(s) = \frac{\lambda^{j-1}(j-1)!}{(s+\lambda)(s+2\lambda)\ldots(s+j\lambda)} \,. \tag{S.124}$$

It is nontrivial to derive, but the inverse Laplace transform is

$$q_j(t) = e^{-\lambda t}(1 - e^{-\lambda t})^{j-1}. \tag{S.125}$$

Problem 3-23.

(a) Since the mean value of $N(t)$ is given by the integral of (3.154), differentiating gives the desired expression.

(b) This also follows easily from the fact that the expectation of the convolution is the convolution of the expectation.

(c) This again follows directly from the interchangeability of expectation and differentiation.

(d) We will just derive the first equation, the second is similar. First note that

$$R_{WX}(t_1,t_2) = E[W(t_1)X(t_2)]$$

$$= E[W(t_1)\int_{-\infty}^{\infty} h(\tau)W(t_2-\tau)\,d\tau]$$

$$= \int_{-\infty}^{\infty} h(\tau)R_{WW}(t_1,t_2-\tau)\,d\tau$$

which we recognize as a convolution in the second argument of the autocorrelation function.

Problem 3-24. Note that

$$E[N(t_1)(N(t_2)-N(t_1))] = E[N(t_1)N(t_2)] - E[N^2(t_1)]. \tag{S.126}$$

Independence implies that the expectation of the product is the product of the expectations:

$$\begin{aligned} E[N(t_1)(N(t_2)-N(t_1))] &= E[N(t_1)]E[N(t_2)-N(t_1)] \\ &= \Lambda(t_1)(\Lambda(t_2)-\Lambda(t_1)). \end{aligned} \tag{S.127}$$

The second moment of a random variable is equal to the square of its mean plus its variance. Since $N(t)$ is a Poisson random variable, we have:

$$E[N^2(t_1)] = \Lambda^2(t_1) + \Lambda(t_1)\ , \tag{S.128}$$

from which the result follows immediately with some minor algebraic manipulation.

Problem 3-25. Noting that $R_{NN}(t_1,t_2)$ is continuous at $t_1 = t_2$, take the derivative first with respect to t_2,

$$R_{N\dot{N}}(t_1,t_2) = \frac{\partial}{\partial t_2}R_{NN}(t_1,t_2) = \begin{cases} \Lambda(t_1)\lambda(t_2), & t_1 < t_2 \\ (1+\Lambda(t_1))\lambda(t_2), & t_1 \ge t_2 \end{cases}. \tag{S.129}$$

Now take the derivative with respect to t_1, first noting that there is a discontinuity of size $\lambda(t_2)$ at $t_1 = t_2$,

$$R_{\dot{N}\dot{N}}(t_1,t_2) = \frac{\partial}{\partial t_1}R_{N\dot{N}}(t_1,t_2) = \lambda(t_1)\lambda(t_2) + \lambda(t_2)\delta(t_1-t_2). \tag{S.130}$$

Finally, we convolve this result with first $h(t_1)$ and then $h(t_2)$ to obtain the autocorrelation of shot noise. First convolving with $h(t_1)$,

$$R_{\dot{N}\dot{N}}(t_1,t_2) * h(t_1) = [\lambda(t_1) * h(t_1)]\lambda(t_2) + \lambda(t_2)h(t_1-t_2) \tag{S.131}$$

and then convolving with $h(t_2)$, we obtain the desired result.

Problem 3-26. Substituting a constant rate into the autocorrelation of (3.218), the autocorrelation is

$$R_{XX}(t_1,t_2) = \lambda^2 H^2(0) + \lambda \int_{-\infty}^{\infty} h(-u)h(t_2 - t_1 - u)\,du \tag{S.132}$$

which is a function of $\tau = t_2 - t_1$ and hence the autocorrelation is

$$R_X(\tau) = \lambda^2 H^2(0) + \lambda h(\tau) * h(\tau). \tag{S.133}$$

Taking the Fourier transform of this expression we get the power spectrum,

$$S_X(f) = \lambda^2 H^2(0)\delta(f) + \lambda\,|H(f)|^2. \tag{S.134}$$

Note that the power spectrum has a d.c. term, corresponding to the expected value of the shot noise, and a term proportional to the magnitude-squared of the filter transfer function (as expected).

Problem 3-27. The expected value of the shot noise is

$$E[X(t)] = \lambda_0 \int_{-\infty}^{t} h(\tau)d\tau \tag{S.135}$$

which is proportional to the step function.

Problem 3-28. If the filter has impulse response $h(t)$ and transfer function

$$H(f) = A(f)e^{j\phi(f)} \tag{S.136}$$

then the response of the filter to λ_0 is $\lambda_0 A(0)$ and the response to the sinusoid is $\lambda_1 A(f_1)\cos(2\pi f_1 t + \phi(f_1))$. The mean value is the sum of these two signals.

Problem 3-29. The outputs are still obviously equally probable, $p(0) = p(1) = 1/2$. From Fig. 3-19 we can calculate $P_{1|1}(z)$,

$$P_{1|1}(z) = \frac{1 - qz^{-1}}{1 - (qz^{-1} + qz^{-1} + p^2 z^{-2}) + q^2 z^{-2}} \tag{S.137}$$

$$S_X^{+}(z) = 0.5 \cdot \frac{1 - qz^{-1}}{(1 - z^{-1})(1 - (1 - 2p)z^{-1})}\ . \tag{S.138}$$

Recognizing that $\mu_X = 1/2$ and subtracting $\frac{0.25}{1 - z^{-1}}$ from the above we get

$$S_X^{+}(z) - \frac{0.25}{1 - z^{-1}} = \frac{0.25}{1 - (1 - 2p)z^{-1}}\ . \tag{S.139}$$

Calculating the two-sided spectrum, we get

$$S_X(z) = \frac{0.25}{1 - (1 - 2p)z^{-1}} + \frac{0.25}{1 - (1 - 2p)z} - 0.25\ , \tag{S.140}$$

which simplifies to the desired result.

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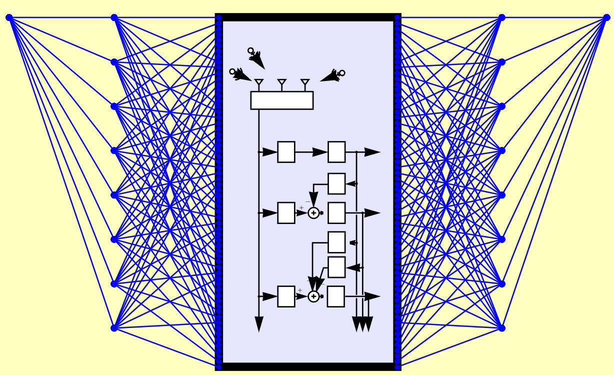

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Chapter 4

Problem 4-1. The entropy is 0.81 bits. The rate is $R = 0.81$ bits/second. There are many possibilities for the coder. Here is one. Pair the coin flips and represent them as follows:

FLIPS	BITS
TT	0
TH	10
HT	110
HH	111

The first occurs with probability 9/16, the second and third with probability 3/16 and the last with probability 1/16. The expected number of bits per flip-pair is therefore 1.67, or 0.844 bits per flip, which is close to the rate of the source, but not equal.

Problem 4-2.

$$H(X) = \frac{1}{2}\log_2(2) + \frac{1}{4}\log_2(4) + \frac{1}{8}\log_2(8) + \frac{1}{8}\log_2(8) = 1.75 \quad . \tag{S.141}$$

$$R = 175 \text{ trials/second} \quad . \tag{S.142}$$

A coder that will work is given by the following table.

outcome	bits
a_1	0
a_2	10
a_3	110
a_4	111

The average number of bits per trial is

$$\frac{1}{2} \cdot 1 + \frac{1}{4} \cdot 2 + \frac{1}{8} \cdot 3 + \frac{1}{8} \cdot 3 = 1.75 \quad . \tag{S.143}$$

Problem 4-3. Suppose that $p_i = p_Y(y_i)$ where the set of y_i for $i \in \{1, 2, \ldots, M\}$ is the alphabet for the random variable Y. Define a new random variable X in terms of Y such that an outcome $Y = y_i$ results in the outcome $X = q_i / p_i$. By Jensen's inequality,

$$E[\log_2 X] \le \log_2 E[X] \tag{S.144}$$

but

$$E[\log_2 X] = \sum_{i=1}^{M} p_i \log_2\left(\frac{q_i}{p_i}\right) = \sum_{i=1}^{M} p_i \log_2 q_i - \sum_{i=1}^{M} p_i \log_2 p_i \tag{S.145}$$

and

$$E[X] = \sum_{i=1}^{M} q_i = \alpha. \tag{S.146}$$

The p-q inequality follows.

Problem 4-4. Let $q_i = 1/K$ in the p-q inequality and the result follows easily.

Problem 4-5. The size of the set over which $\mathbf{X}$ has positive probability is less than or equal to 2^n, so the result follows immediately from Exercise 4-1.

Problem 4-6.

(a) It is easy to show that the input and output of this channel are statistically independent, and from this that

$$H(X|Y) = H(X) \ , \qquad H(Y|X) = H(Y) \ , \tag{S.147}$$

$$I(X,Y) = 0 \ . \tag{S.148}$$

(b) Since the mutual information is zero independent of the input probability distribution, the capacity is also zero.

Problem 4-7.

(a) For this noiseless channel, the output is equal to the input, so

$$H(X|Y) = 0 \ , \qquad H(Y|X) = 0 \ , \tag{S.149}$$

$$I(X,Y) = H(X) = H(Y) \ . \tag{S.150}$$

(b) The capacity is the maximum of $H(X)$ over the input distribution, which is one bit because the input has alphabet size of two.

Problem 4-8.

(a) We get

$$I(X,Y) = H(Y) + p\log_2 p + (1-p)\log_2(1-p) \ . \tag{S.151}$$

(b) Capacity is achieved when $H(Y)$ is maximized. By direct calculation,

$$H(Y) = -p\log_2 p - (1-p)\log_2(1-p) - q(1-p)\log_2 q - (1-q)(1-p)\log_2(1-q) \ , \tag{S.152}$$

where q is the probability of the first input and $(1-q)$ is the probability of the second. This quantity is maximized when the inputs are equally likely ($q = 1/2$), and the capacity is

$$C_s = 1 - p \ . \tag{S.153}$$

The center output is called an *erasure*, and tells us nothing about what the channel input is, so it is not surprising that the capacity approaches zero as $p \to 1$.

Problem 4-9.

(a) This follows from

$$\sum_{i=1}^{M} p_i \log\left(\frac{q_i}{p_i}\right) \le \sum_{i=1}^{M} p_i\left(\frac{q_i}{p_i} - 1\right) = 0 \tag{S.154}$$

where we have used the inequality $\log(x) \le x - 1$.

(b) Substitute a uniform distribution $q_i = 1/K$, and the bound follows immediately.

Problem 4-10.

(a) The channel has binary input and output, so all we need to show is symmetry. To do this, we solve part b.

(b) The number of transitions from 0 to 1 or vice versa is binomially distributed. As long as there are an even number of such crossovers, then the output of the channel will be the same as the input. The probability of this occurring is

$$p_{Y|X}(0|0) = p_{Y|X}(1|1) = \sum_{\substack{m=0 \\ \text{even}}}^{L} K_m (1-p)^{L-m} p^m \tag{S.155}$$

where

$$K_m = \binom{L}{m} = \frac{L!}{m!(L-m)!} \quad . \tag{S.156}$$

Similarly, a channel error occurs if there are an odd number of crossovers, which occurs with probability

$$p_{Y|X}(1|0) = p_{Y|X}(0|1) = \sum_{\substack{m=1 \\ \text{odd}}}^{L} K_m (1-p)^{L-m} p^m \quad . \tag{S.157}$$

(c)

$$p_{X|Y}(0|0) = p_{X|Y}(1|0) = p_{X|Y}(0|1) = p_{X|Y}(1|1) \to 1/2 \text{ as } \quad L \to \infty \, . \tag{S.158}$$

Problem 4-11. Suppose X has distribution p_i and Y has distribution q_i. Then

$$H(\text{r}X) = -\sum_{i=1}^{K} p_i \log_2 p_i \le -\sum_{i=1}^{K} p_i \log_2 q_i \quad , \tag{S.159}$$

where the inequality follows from Problem 4-9. Meanwhile,

$$\begin{aligned} H(Y) &= -\sum_{i=1}^{K} q_i \log_2 q_i \\ &= -(p_1 - \delta)\log_2 q_1 - (p_2 + \delta)\log_2 q_2 - \sum_{i=3}^{K} p_i \log_2 q_i \\ &= -\sum_{i=1}^{K} p_i \log_2 q_i + [\delta \log_2 q_1 - \delta \log_2 q_2] \quad . \end{aligned} \tag{S.160}$$

Since $p_1 > p_2$, we have that $q_1 > q_2$, and the term in brackets in nonnegative, so by comparing with (S.159) we get that $H(Y) \ge H(X)$.

Problem 4-12.

(a)

$$H(X) = -\int_{-a}^{a} \frac{1}{2a} \log_2\left(\frac{1}{2a}\right) dx = -\log_2\left(\frac{1}{2a}\right) . \tag{S.161}$$

(b) The variance of X is

$$\sigma^2 = E[X^2] = \int_{-a}^{a} \frac{1}{2a} x^2 dx = \frac{1}{3} a^2 . \tag{S.162}$$

Hence we can write

$$a = \sqrt{3}\,\sigma \tag{S.163}$$

and

$$H(X) = \log_2(2\sqrt{3}\,\sigma) . \tag{S.164}$$

The entropy of a Gaussian is given by (4.21),

$$H(Y) = \frac{1}{2} \log_2(2\pi e \sigma^2) = \log_2(\sigma\sqrt{2\pi e}) . \tag{S.165}$$

Since $2\sqrt{3} < \sqrt{2\pi e}$, then $H(X) < H(Y)$.

Problem 4-13. Let the random vector Z denote the ordered pair (X, Y). The set of all possible pairs of outcomes of Z is in $\Omega_X \times \Omega_Y$. Number the possible outcomes (in any order) from 1 through M, where M is the size of Ω_X multiplied by the size of Ω_Y. Then let z_i denote a particular pair of outcomes (x, y), where $1 \le i \le M$. Then define

$$p_i = p_{X,Y}(x, y) \tag{S.166}$$

Further, define

$$q_i = p_X(x) p_Y(y) . \tag{S.167}$$

These p_i and q_i satisfy the constraints of the p-q inequality. The p-q inequality then yields

$$-\ I(X,Y) = -\sum_{i=1}^{M} p_i \log_2\left(\frac{q_i}{p_i}\right) \le \log_2 \sum_{i=1}^{M} p_i\left(\frac{q_i}{p_i}\right) = \alpha. \tag{S.168}$$

Hence,

$$\begin{aligned} I(X,Y) \ge \alpha &= \log_2\left(\sum_{x \in \Omega_X} \sum_{x \in \Omega_Y} p_X(x) p_Y(y)\right) \\ &= \log_2\left(\left(\sum_{x \in \Omega_X} p_X(x)\right)\left(\sum_{x \in \Omega_Y} p_Y(y)\right)\right) \\ &= \log_2 1 = 0. \end{aligned} \tag{S.169}$$

The second and third inequalities follow easily. The inequalities are equalities when X and Y are independent.

Problem 4-14. This follows easily with repeated application of the definition of conditional probability.

Problem 4-15. Using the inequality $\log_e(x) \le (x-1)$,

$$C \le \frac{N}{2} \cdot \log_e 2 \cdot \frac{\sigma_x^2}{N\sigma^2} = \log_2 \sqrt{e} \cdot \frac{\sigma_x^2}{\sigma^2} \quad . \tag{S.170}$$

Thus, as the number of degrees of freedom increases, the capacity approaches a constant. As we increase N we have in effect more parallel channels to transmit over, and hence the factor of N. However, since the total input power is constrained to ${\sigma_x}^2$, the transmitter is forced to reduce the power in each parallel channel, and hence the SNR on each channel decreases. In the limit, these two effects precisely balance one another.

Problem 4-16. The implication of (4.47) is that for any set of outcomes $x_1,\ldots, x_n$,

$$\frac{1}{n}(x_1 + \ldots + x_n) \approx E[X] \ , \tag{S.171}$$

with high probability. Let $X_i = \log_2 Y_i$. Then

$$E[\log_2 Y] \approx \frac{1}{n}\log_2(y_1 \ldots y_n) \tag{S.172}$$

with high probability. The result follows from this.

Problem 4-17. Define the random variable Y to have value 5 with probability $1/6$ and value $1/5$ with probability $5/6$. Then the money left after playing the game n times is

$$M_n = 100 \prod_{i=1}^{n} Y_i \tag{S.173}$$

where Y_i is a sequence of independent trials of the random variable Y. Now, because of independence,

$$E[M_n] = 100 \prod_{i=1}^{n} E[Y_i] = 100. \tag{S.174}$$

Surprisingly, this does not imply that the game is fair. From (4.39), with high probability, a set of outcomes $y_1,\ldots,y_n$ satisfies

$$M_n = 100 \prod_{i=1}^{n} Y_i \approx [2^{E[\log_2 Y]}]^n \approx [0.34]^n. \tag{S.175}$$

This certainly goes to zero as n gets large. I wouldn't play the game, at least not repeatedly.

Problem 4-18. The better estimate is (b). The argument is similar to that in the solution to Problem 4-17.

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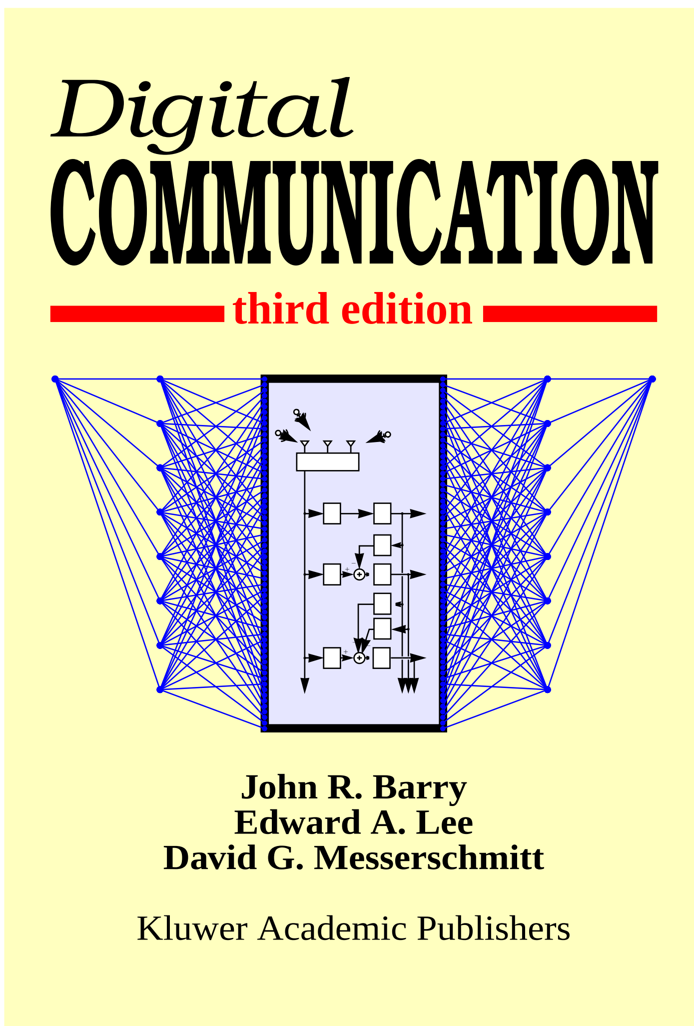


Chapter 5

Problem 5-1. From section 5.1.2, the channel output is a PAM signal whose pulse shape $h(t)$ is related to the transmit pulse shape $b(t)$ and the channel impulse response $b(t)$ by:

$$H(f) = G(f)B(f). \tag{S.176}$$

Using the result of Appendix 3-A, we can write the power spectrum of the received signal as

$$S_R(f) = \frac{1}{T}\,|H(f)|^2 S_A(e^{j2\pi fT}) = \frac{E_a}{T}\,|G(f)B(f)|^2 = \frac{E_a}{T}\,|G(f)|^2 B(f). \tag{S.177}$$

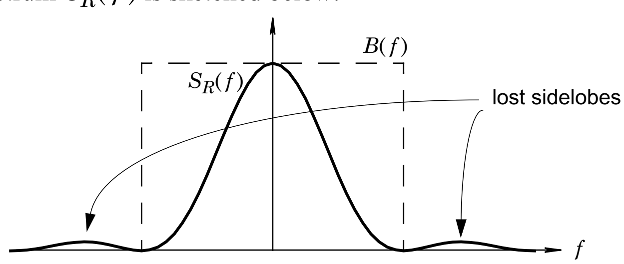
Using Appendix 2-A,

$$G(f) = Te^{-j\pi fT}\frac{\sin(\pi fT)}{\pi fT}, \tag{S.178}$$

so

$$|G(f)|^2 = \frac{\sin^2(\pi fT)}{(\pi f)^2}. \tag{S.179}$$

The power spectrum $S_R(f)$ is sketched below:



The distortion is due to the loss of the sidelobes of $G(f)$ when it is multiplied by $B(f)$.

Problem 5-2. From Appendix 3-A, the transmit power spectrum for baseband PAM is

$$S_S(f) = \frac{1}{T}\,|G(f)|^2 S_a(e^{j\theta}) = \frac{1}{T}\,|G(f)|^2. \tag{S.180}$$

Integrating the power spectrum yields the power:

$$P = \frac{1}{T}\int_{-\infty}^{\infty} |G(f)|^2\, df = \frac{E_g}{T}, \tag{S.181}$$

where E_g is the energy of the pulse shape $g(t)$. (The relationship $P = E_g/T$ makes intuitive sense, since power is energy per unit time.) But:

$$E_g = 2\int_0^{\infty} |G(f)|^2\,df$$

$$= 2\int_0^{\left(\frac{1-\alpha}{2T}\right)} T^2 df + 2\int_{\left(\frac{1-\alpha}{2T}\right)}^{\left(\frac{1+\alpha}{2T}\right)} \left(\frac{T}{2}\left(1+\cos\left(\pi\left(f-\left(\frac{1-\alpha}{2T}\right)\right)T/\alpha\right)\right)\right)^2 df. \quad \text{(S.182)}$$

Let $\nu = f - \left(\frac{1-\alpha}{2T}\right)$:

$$E_g = (1-\alpha)T + \frac{T^2}{2}\int_0^{\frac{\alpha}{T}} (1+\cos(\pi\nu T/\alpha))^2\,d\nu$$

$$= (1-\alpha)T + \frac{T^2}{2}\int_0^{\frac{\alpha}{T}} (1+2\cos(\pi\nu T/\alpha)+\cos^2(\pi\nu T/\alpha))d\nu$$

$$= (1-\alpha)T + \frac{T^2}{2}\left(\frac{\alpha}{T} + 0 + \frac{\alpha}{2T}\right)$$

$$= (1-\alpha/4)T\,. \quad \text{(S.183)}$$

Therefore:

$$P = \frac{E_g}{T} = 1 - \frac{\alpha}{4}, \quad \text{(S.184)}$$

which is independent of T.

Problem 5-3. The minimum bandwidth pulse has bandwidth $1/(2T)$ Hz, where $1/T$ is the symbol rate. The bandwidth of a pulse with 50% excess bandwidth is $1.5/(2T)$ Hz. We require that

$$1.5/(2T) \le 1500\,, \quad \text{(S.185)}$$

which implies that the maximum symbol rate is $\frac{1}{T} = 2000$ symbols per second.

Problem 5-4.

(a) The input bit sequence is white, so that $R_x(m) = \delta_m$ and $S_x(e^{j\theta}) = 1$. Further,

$$a_k = x_k - x_{k-1}\ . \quad \text{(S.186)}$$

Taking Z-transforms,

$$A(z) = X(z)(1-z^{-1})\ . \quad \text{(S.187)}$$

Define

$$H(z) = \frac{A(z)}{X(z)} = 1 - z^{-1} \quad \text{(S.188)}$$

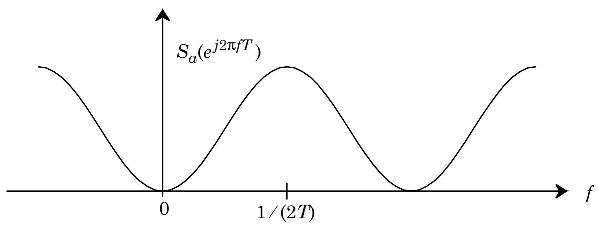
so that

$$H(e^{j2\pi fT}) = 1 - e^{-j2\pi fT} \quad . \tag{S.189}$$

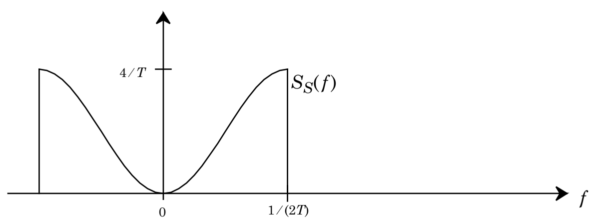
Hence

$$\begin{aligned} S_a(e^{j2\pi fT}) &= S_x(e^{j2\pi fT})\,|H(e^{j2\pi fT})|^2 \\ &= |1 - e^{-j2\pi fT}|^2 \\ &= 2 - 2\cos(2\pi fT) \quad . \end{aligned} \tag{S.190}$$

The sketch is shown in the following figure.



(b) Here is a well-labeled, careful sketch:



(c) Starting with:

$$\begin{aligned} s(t) &= \sum_{m=-\infty}^{\infty} (x_m - x_{m-1}) g(t - mT) \\ &= \sum_{m=-\infty}^{\infty} X_m g(t - mT) - \sum_{m=-\infty}^{\infty} X_m g(t - mT - T) \\ &= \sum_{m=-\infty}^{\infty} X_m [g(t - mT) - g(t - mT - T)] \quad . \end{aligned} \tag{S.191}$$

So

$$h(t) = g(t) - g(t - T) \tag{S.192}$$

and

$$H(f) = (1 - e^{-j2\pi fT}) G(f) \quad . \tag{S.193}$$

Note that $H(f) = 0$ for all $f = m/T$, so the Nyquist criterion can never be satisfied, regardless of $G(f)$. Furthermore,

$$P(f) = (1 - e^{-j2\pi fT}) G(f) F(f) \tag{S.194}$$

does not satisfy the Nyquist criterion for any $G(f)$ and $F(f)$ that are finite for all f. (It turns out that ISI can be eliminated using a receive filter that has infinite gain at DC, but this filter is not stable.)

Problem 5-5. The only zero excess-bandwidth pulse satisfying the Nyquist criterion is (5.6). But since the pulse falls off as $1/t$, the worst case transmitted symbol sequence will result in infinite intersymbol interference for any sampling phase other than the ideal phase (where the intersymbol interference is zero).

Consider binary PAM signaling with a zero excess-bandwidth raised-cosine pulse shape, which is an ordinary sinc pulse. We can show that if the timing phase is off from the ideal by any amount, the data cannot be recovered. The pulse shape is given by

$$g(t) = \frac{\sin(\pi t/T)}{\pi t/T} \tag{S.195}$$

and the PAM signal by

$$\begin{aligned} r(t) &= \sum_{m=-\infty}^{\infty} a_m g(t-mT) \\ &= \sum_{m=-\infty}^{\infty} a_m \frac{\sin(\pi(t-mT)/T)}{\pi(t-mT)/T} \quad . \end{aligned} \tag{S.196}$$

Sampling with exactly the correct timing phase, we get for any integer k

$$r(kT) = a_k \quad . \tag{S.197}$$

To see what happens if the timing phase is slightly off, we rewrite the signal using trigonometric properties of the sine function,

$$r(t) = \frac{T}{\pi}\sin\left(\frac{\pi t}{T}\right)\sum_{m=-\infty}^{\infty} a_m \frac{(-1)^m}{(t-mT)} \quad . \tag{S.198}$$

Evaluating at time $t = \varepsilon \approx 0$, and using the approximation $\sin(x) \approx x$ for small x, gives:

$$r(\varepsilon) \approx \varepsilon \sum_{m=-\infty}^{\infty} a_m \frac{(-1)^m}{(\varepsilon - mT)} \approx a_0 + \sum_{m \neq 0} a_m \frac{(-1)^m}{-mT} \quad . \tag{S.199}$$

This can be far from the desired value a_0. In particular, with the unfortunate symbol sequence $a_m = (-1)^{m+u_m}$, where u_m is the unit step, the above reduces to:

$$r(\varepsilon) \approx a_0 + \frac{2}{T}\sum_{m \neq 0} \frac{1}{m}, \tag{S.200}$$

which is not even finite. Since the interference from neighboring symbols may not even be finite for any arbitrarily small error in the timing phase, the width of the eye is zero. Consequently, binary antipodal excess bandwidth *must* be larger than zero. (A different conclusion might arise from a probabilistic exploration of this problem, since the probability of sequences such as $(-1)^{m+u_m}$ are asymptotically zero.)

Problem 5-6.

(a)

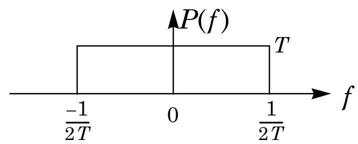
$$E[|a_k|^2] = R_a(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_a(e^{j\theta})d\theta = 1 \quad . \tag{S.201}$$

(b) From Appendix 3-A, the power spectrum for a baseband PAM signal is

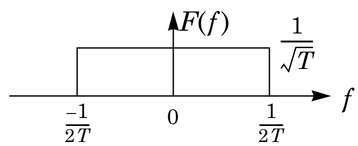
$$S_S(f) = \frac{1}{T}|G(f)|^2 S_a(e^{j\theta}) = \frac{1}{T}|G(f)|^2 = 1 \, , \tag{S.202}$$

which is independent of T.

(c) We need to find $F(f)$ so that $P(f) = G(f)F(f)$ is as sketched below:



Since $G(f) = \sqrt{T}$, we can write $F(f) = P(f)/\sqrt{T}$, so that the desired received filter has the following frequency response:



Yes, the pulse satisfies the Nyquist criterion.

Let us find a general solution that applies to both parts (d) and (e). From (5.52) and (5.53), the signal-to-noise ratio is defined by:

$$\text{SNR} = \frac{\left(\int g(t)f(-t)dt\right)^2}{E_f} \cdot \frac{E_a}{N_0/2} \, , \tag{S.203}$$

where we changed the noise power spectral density to reflect the fact that this problem concerns a real, baseband system. In the above expression, $E_a = \text{E}[a_k{}^2]$ is the energy in the symbols, and E_f is the energy in the receive filter $f(t)$. The integral in the numerator is easily recognized as the convolution of $g(t)$ and $f(t)$, which is $p(t)$, evaluated at time zero:

$$\int_{-\infty}^{\infty} g(t)p(-t)dt = p(0). \tag{S.204}$$

Hence, the SNR reduces to:

$$\text{SNR} = \frac{2p^2(0)E_a}{E_f N_0} \quad . \tag{S.205}$$

(d) When $F(f)$ is as given in part (c), its energy is clearly unity: $E_f = 1$. Furthermore, since $p(0) = 1$ and $E_a = 1$, the SNR from (S.205) reduces to:

$$\text{SNR} = \frac{2}{N_0} . \tag{S.206}$$

(e) We still have $E_a = 1$, and it is easy to see that $p(0) = 1$. It remains only to find the energy E_f. Since $G(f) = \sqrt{T}$, it follows that $F(f) = P(f)/\sqrt{T}$ and thus:

$$\begin{aligned} E_f &= \int_{-\infty}^{\infty} |F(f)|^2 df \\ &= \frac{1}{T}\int_{-\infty}^{\infty} |P(f)|^2 df \\ &= \frac{2}{T}\int_0^{1/T} T^2(1 - fT)^2 df \\ &= \frac{2}{T}\int_0^{1/T} T^2(fT)^2 df \\ &= 2T^3\int_0^{1/T} f^2 df \\ &= \frac{2T^3}{3T^3} = \frac{2}{3} . \end{aligned} \tag{S.207}$$

Plugging into (S.205), the SNR reduces to:

$$\text{SNR} = \frac{3}{N_0} . \tag{S.208}$$

This is better than in part (d), even though the receive filter bandwidth is greater.

Problem 5-7.

(a) Yes this pulse satisfies the Nyquist criterion; we recognize it as a shifted version of the Nqyuist pulse from Fig. 5-3(b). In the time domain, this pulse $g(t)$ is related to that of Fig. 5-3(b), call it $y(t)$, by:

$$g(t) = y(t)e^{j2\pi t/T} . \tag{S.209}$$

Multiplying a Nyquist pulse by anything that is nonzero at time zero will necessarily result in another Nyquist pulse.

Yes, $\text{Re}\{g(t)\}$ also satisfies the Nyquist criterion, because from (S.209):

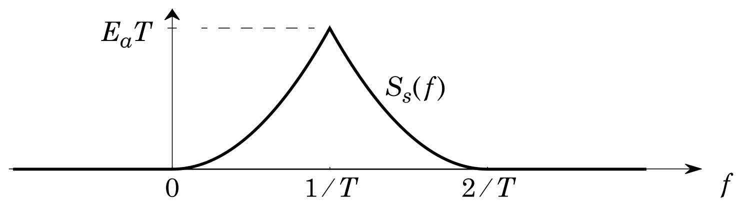
$$\text{Re}\{g(t)\} = y(t)\cos(2\pi t/T) . \tag{S.210}$$

Again, multiplying a Nyquist pulse by anything that is nonzero at time zero will necessarily result in another Nyquist pulse.

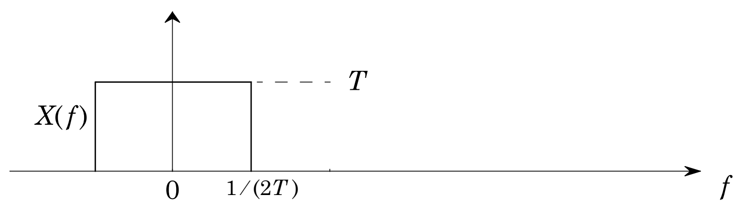
(b) From Appendix 3-A we get

$$S_s(f) = \frac{E_a}{T} |G(f)|^2 \tag{S.211}$$

which is sketched below.



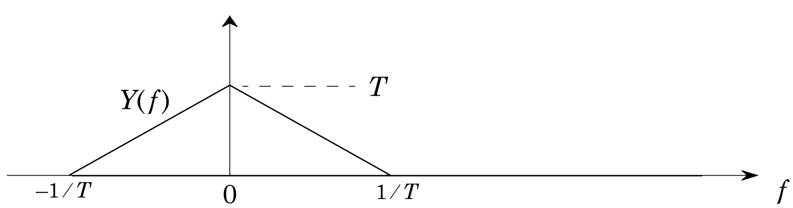
(c) Let $x(t) = \sin(\pi t/T)/(\pi t/T)$ be the ideal zero-excess-bandwidth pulse, with Fourier transform as shown below:



Convolving two rectangles gives a triangle:

$$Y(f) = X(f) * X(f) \ , \tag{S.212}$$

as sketched below:



Observe that

$$G(f) = Y(f - 1/T) \ . \tag{S.213}$$

In the time-domain, from Appendix 2-A, these previous two equations imply that:

$$y(t) = [x(t)]^2 \tag{S.214}$$

and

$$g(t) = y(t)e^{j2\pi t/T} \ . \tag{S.215}$$

Hence

$$g(t) = e^{j2\pi t/T}\left(\frac{\sin(\pi t/T)}{\pi t/T}\right)^2 \ . \tag{S.216}$$

Problem 5-8.

From (5.52) and (5.53), the signal-to-noise ratio is defined by:

$$\mathrm{SNR} = \frac{\left|\int g(t)f(-t)dt\right|^2}{E_f} \cdot \frac{E_a}{N_0/2}, \tag{S.217}$$

where we changed the noise power spectral density to reflect the fact that this problem concerns a real, baseband system. In the above expression, $E_a = \mathrm{E}[a_k^2]$ is the energy in the symbols, and E_f is the energy in the receive filter $f(t)$. The integral in the numerator is easily recognized as the convolution of $g(t)$ and $f(t)$, which is $p(t)$, evaluated at time zero:

$$\int_{-\infty}^{\infty} g(t)f(-t)dt = p(0). \tag{S.218}$$

Hence, the SNR can also be written as:

$$\mathrm{SNR} = \frac{p^2(0)E_a}{E_f N_0/2}. \tag{S.219}$$

In this problem the overall (equalized) pulse $p(t)$ is the raised-cosine pulse of (5.8), which satisfies $p(0) = 1$ by design, so that:

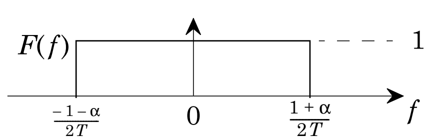
$$\mathrm{SNR} = \frac{E_a}{E_f N_0/2}. \tag{S.220}$$

This equation makes intuitive sense: When the overall pulse is Nyquist with $p(0) = 1$, sampling the output yields the transmitted symbol plus sampled noise. The numerator above is the power of the signal term, and the denominator is the power of the sampled noise.

(a) When the transmitted pulse is an impulse, $G(f) = 1$, and under the assumption that the channel is distortionless, the receive filter and the overall pulse shape are one in the same, $F(f) = P(f)$. As we saw in (S.183) for Problem 5-2, the energy of the raised-cosine pulse is $E_p = E_f = (1 - \alpha/4)T$. Therefore, the SNR is:

$$\mathrm{SNR} = \frac{2E_a}{\left(1 - \frac{\alpha}{4}\right)TN_0}. \tag{S.221}$$

(b) The receive filter required in this case is nothing but an ideal low-pass filter, as sketched below:



The energy of this filter is clearly $E_f = (1 + \alpha)/T$. Therefore, the SNR reduces to:

$$\mathrm{SNR} = \frac{2E_a T}{(1 + \alpha)N_0}. \tag{S.222}$$

(c) (Contrary to the comment, this is not tedious.) The receive filter in this case is also a square-root raised-cosine pulse, $F(f) = P(f)^{1/2}$. Therefore, the energy in the receive filter is easy to calculate:

$$\begin{aligned} E_f &= \int_{-\infty}^{\infty} |F(f)|^2 df \\ &= \int_{-\infty}^{\infty} P(f) df \\ &= p(0) \\ &= 1. \end{aligned} \tag{S.223}$$

So the SNR in this case is:

$$\text{SNR} = \frac{E_a}{E_f N_0/2} = \frac{E_a}{N_0/2}\,. \tag{S.224}$$

Problem 5-9.

(a) 64 kb/s are required, so with binary antipodal signaling this is 64,000 symbols per second, so the minimum bandwidth is 32 kHz.

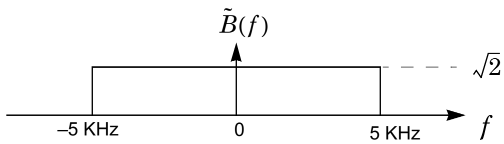
(b) 64 kHz.

(c) 32 kHz.

(d) 16 kHz.

Problem 5-10.

(a) The answer depends on two things: the passband gain and the nominal carrier frequency. Assume the passband gain is unity, and assume the nominal carrier frequency is the midpoint of the pass band. Then the Fourier transform of the complex envelope of the channel can be written down by inspection:



Therefore, the complex envelope is the corresponding sinc function:

$$\tilde{b}(t) = \sqrt{2}\frac{\sin(10000\pi t)}{\pi t}\,. \tag{S.225}$$

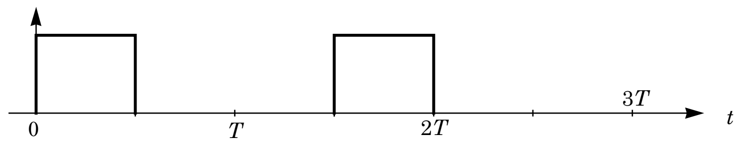
(b) The answers are respectively 20 Kb/s, 6667 b/s, 20 Kb/s, and 40 kb/s. In the latter case, the receive filter should be properly designed to meet the Nyquist criterion.

(c) A reasonable transmit filter is $g(t) = \sqrt{2}\tilde{b}(t)$. This makes full use of the band, but doesn't transmit any energy outside of the pass band (which would get rejected by the channel anway). In this case, the Fourier transform of the received pulse shape is $H(f) = \tilde{B}(f)^2$. A *matched* receive filter will yield an overall pulse shape of $P(f) = |H(f)|^2 = \tilde{B}(f)^4$ that is itself a Nyquist pulse for the 10 KHz symbol rate.

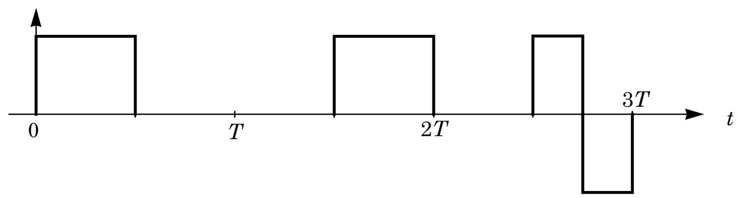
Problem 5-11. Use ROMs with N bit addresses, two outputs, each with $2^{\lceil \frac{N}{2} \rceil}$ bits.

Problem 5-12.

(a) Here is a Nyquist pulse of duration $2T$:



(b) Here is a Nyquist pulse of duration $3T$:



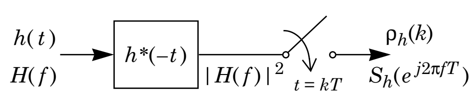
Problem 5-13.

(a)

$$\rho_h^*(-k) = \int_{-\infty}^{\infty} h^*(t)h(t+kT)\ dt = \int_{-\infty}^{\infty} h(u)h^*(u-kT)\ du\ , \tag{S.226}$$

where the change of variables $u = t + kT$ has been made.

(b) The sampled autocorrelation function is defined by $\rho_h(k) = \int_{-\infty}^{\infty} h(t)h^*(t-kT)dt$. Recognizing this integral as the convolution of $h(t)$ with $h^*(-t)$, evaluated at time kT, we can view $\rho_h(k)$ as the output of the sampled matched filter when the received pulse $h(t)$ is the input, as shown below:



The Fourier transform of the signal before the sampler is $|H(f)|^2$. Therefore, the Fourier transform of the signal after the sampler, which is the folded spectrum, is given by:

$$S_h(e^{j2\pi fT}) = \frac{1}{T}\sum_{m=-\infty}^{\infty} |H(f-m/T)|^2. \tag{S.227}$$

It is nonnegative and real because it is the sum of things that are nonnegative and real.

Problem 5-14. The folded spectrum Z-transform can be broken down into two summations, one for positive time and one for negative time:

$$S_h(z) = \sum_{k=-\infty}^{\infty} \rho_h(k) z^{-k}$$

$$= \sum_{k=0}^{\infty} \rho_h(k) z^{-k} + \sum_{k=-\infty}^{0} \rho_h(k) z^{-k} - \rho_h(0) . \tag{S.228}$$

The first summation is $S_{h,+}(z)$. The second summation can be rewritten as:

$$\sum_{k=0}^{\infty} \rho_h(-k) z^k = \sum_{k=0}^{\infty} \rho_h{}^*(k) z^k = S_{h,+}{}^*(1/z^*) . \tag{S.229}$$

Problem 5-15.

(a)

$$\begin{aligned}
\rho_h(k) &= \int_{-\infty}^{\infty} \sum_{k=0}^{K} f_m h_0(t - mT) \cdot \sum_{n=0}^{K} f_n{}^* h_0{}^*(t - nT - kT)\, dt \\
&= \sum_{k=0}^{K} f_m \sum_{n=0}^{K} f_n{}^* \int_{-\infty}^{\infty} h_0(t - mT) h_0{}^*(t - nT - kT)\, dt \\
&= \sum_{k=0}^{K} f_m \sum_{n=0}^{K} f_n{}^* E_0 \delta_{m-k-n} \\
&= E_0 \sum_{m=0}^{K} f_m f_{m-k}{}^* \ .
\end{aligned} \tag{S.230}$$

Taking the Z transform of this results directly in (5.138).

(b) The pulse energy is $E_h = \int_{-\infty}^{\infty} |h(t)|^2 dt = \rho_h(0)$, which from part (a) gives:

$$E_h = E_0 \sum_{m=0}^{K} |f_m|^2 \ . \tag{S.231}$$

Problem 5-16.

(a) $Q(2) = 0.0228$, which is about 44 times larger than $Q^2(2) = 5.176 \times 10^{-4}$.

(b) $Q(4) = 3.167 \times 10^{-5}$, which is about 31574 times larger than $Q^2(4) = 10^{-9}$.

Neglecting Q^2 relative to Q is clearly valid in these cases.

Problem 5-17.

The first probability is $Q(\sqrt{2}/0.5) = 0.0023$, which is about 74 times bigger than the second probability $Q(2/0.5) = 3.17 \times 10^{-5}$.

Problem 5-18.

As discussed in Example 5-37, the probability of error for the four innermost points is

$$\begin{aligned}
\Pr[\text{error} \mid a \text{ on the inside}] &= [1 - 2Q(d/2\sigma)]^2 \\
&= 4Q(d/2\sigma) - 4Q^2(d/2\sigma) .
\end{aligned} \tag{S.232}$$

Similarly, the probability of error for the four corner points is

$$\begin{aligned}
\Pr[\text{error} \mid a \text{ in the corner}] &= [1 - Q(d/2\sigma)]^2 \\
&= 2Q(d/2\sigma) - Q^2(d/2\sigma) .
\end{aligned} \tag{S.233}$$

The probability of error for the eight edge points is:

$$\Pr[\text{error} \,|\, a \text{ on edge}] = [1 - 2Q(d/2\sigma)][1 - Q(d/2\sigma)]$$
$$= 3Q(d/2\sigma) - 2Q^2(d/2\sigma). \tag{S.234}$$

The result follows by adding these terms, weighted by their probabilities:

$$\begin{aligned}\Pr[\text{error}] = \;& \Pr[a \text{ on inside}]\Pr[\text{error} \,|\, a \text{ on the inside}] \\ &+ \Pr[a \text{ on corner}]\Pr[\text{error} \,|\, a \text{ on a corner}] \\ &+ \Pr[a \text{ on ege }]\Pr[\text{error} \,|\, a \text{ on an edge}]\end{aligned} \tag{S.235}$$

$$\begin{aligned}&= \tfrac{4}{16}\left(4Q(d/2\sigma) - 4Q^2(d/2\sigma)\right) \\ &+ \tfrac{4}{16}\left(2Q(d/2\sigma) - Q^2(d/2\sigma)\right) \\ &+ \tfrac{8}{16}\left(3Q(d/2\sigma) - 2Q^2(d/2\sigma)\right)\end{aligned} \tag{S.236}$$

$$= 3Q(d/2\sigma) - 2.25Q^2(d/2\sigma))\ . \tag{S.237}$$

Problem 5-19. We could proceed as in Problem 5-18, which is straightforward, but instead let us take a different route. The constraint that $M = 2^b$ for b even is equivalent to a constraint that $\sqrt{M}$ be a power of 2. In this case, M-ary QAM can be viewed as a *pair* of *real* $\sqrt{M}$ -ary *PAM* signals in parallel, one for the in-phase component and another for the quadrature.

To proceed we need to know the error probability of L-ary PAM. We've already done this in (5.104) for the special case of 4-ary PAM, but we need to generalize to arbitrary L that is a power of two. Let $z = a + n$, where a is uniformly distributed over the real PAM alphabet $\{\pm c, \pm 3c, \ldots \pm(L-1)c\}$, where $c = d/2$ is related to the alphabet energy E_a by $c = \sqrt{\frac{3E_a}{L^2-1}}$, and let $n \sim \mathcal{N}(0, \sigma^2 = N_0/(2E_h))$ be independent real noise. Then:

$$\begin{aligned}P_e^{\text{PAM}}(L, E/N_0) &= \Pr[\text{error for } L\text{-ary PAM}] \\ &= \Pr[a \text{ on end}]\Pr[\text{error} \,|\, a \text{ on end}] \\ &+ \Pr[a \text{ not on end}]\Pr[\text{error} \,|\, a \text{ not on end}]\end{aligned} \tag{S.238}$$

$$\begin{aligned}&= \frac{2}{L}\left(Q(d/2\sigma)\right) + \frac{L-2}{L}\left(2Q(d/2\sigma)\right) \\ &= 2\left(1 - \frac{1}{L}\right)Q(c/\sigma) \\ &= 2\left(1 - \frac{1}{L}\right)Q\left(\sqrt{\frac{6E/N_0}{L^2-1}}\right),\end{aligned} \tag{S.239}$$

where $E = E_a E_h$ is the average received energy per pulse. Observe that this reduces to (5.104) when $L = 4$.

Now back to the view that QAM is a pair of real $\sqrt{M}$ -ary PAM signals in parallel. When the QAM alphabet has energy E_a, the energy of each of the $\sqrt{M}$ -ary PAM alphabets is $E_a/2$. The complex QAM symbol is decoded correctly if and only if both the real and imaginary symbols are decoded correctly. Therefore:

$$
\begin{aligned}
P_e^{\mathrm{QAM}}(M, E/N_0) &= \Pr[\text{error for } M\text{-ary QAM}] \\
&= 1 - \Pr[\text{correct for } \sqrt{M}\text{-ary PAM}]^2 \\
&= 1 - \Big(1 - P_e^{\mathrm{PAM}}(\sqrt{M}, \tfrac{1}{2}E/N_0)\Big)^2 && \text{(S.240)} \\
&= 2P_e^{\mathrm{PAM}}(\sqrt{M}, \tfrac{1}{2}E/N_0) - P_e^{\mathrm{PAM}}(\sqrt{M}, \tfrac{1}{2}E/N_0)^2 && \text{(S.241)} \\
&= 4\Big(1 - \frac{1}{\sqrt{M}}\Big)Q\Big(\sqrt{\frac{3E/N_0}{M-1}}\Big) - \Big(2\Big(1 - \frac{1}{\sqrt{M}}\Big)Q\Big(\sqrt{\frac{3E/N_0}{M-1}}\Big)\Big)^2 && \text{(S.242)} \\
&= 4\Big(1 - \frac{1}{\sqrt{M}}\Big)Q\Big(\sqrt{\frac{3E/N_0}{M-1}}\Big) - 4\Big(1 - \frac{1}{\sqrt{M}}\Big)^2 Q^2\Big(\sqrt{\frac{3E/N_0}{M-1}}\Big). && \text{(S.243)}
\end{aligned}
$$

Note the typo in the text: The second occurence of $\Big(1 - \frac{1}{\sqrt{M}}\Big)$ is supposed to be squared.

Problem 5-20.

(a) There are 4 symbols at distance $d = 2c$, 4 symbols at distance $d = 2\sqrt{2}\,c$, 2 symbols at distance $d = 4c$, 4 symbols at distance $d = 2\sqrt{5}\,c$, and 1 symbol at distance $d = 4\sqrt{2}\,c$, so the union bound is

$$
\begin{aligned}
Pr[\text{symbol error} \,|\, a_k = c + jc] &\le 4Q(c/\sigma) + 4Q(\sqrt{2}\,c/\sigma) + 2Q(2c/\sigma) \\
&\qquad + 4Q(\sqrt{5}\,c/\sigma) + Q(2\sqrt{2}\,c/\sigma) \\
&\approx 4Q(c/\sigma)\,. && \text{(S.244)}
\end{aligned}
$$

(b) There are 2 symbols at distance $d = 2$, 2 symbols at distance $d = \sqrt{5}$, 2 symbols at distance $d = 2\sqrt{2}$, 4 symbols at distance $d = \sqrt{17}$, 2 symbols at distance $d = 2\sqrt{5}$, 1 symbol at distance $d = 4\sqrt{2}$, and 2 symbol at distance $d = \sqrt{37}$, so the union bound is

$$
\begin{aligned}
\Pr[\text{symbol error} \,|\, a_k = 1 + j\,] &\le 2Q(1/\sigma) + 2Q(\sqrt{5}/2\sigma) + 2Q(\sqrt{2}/\sigma) \\
&\qquad + 4Q(\sqrt{17}/2\sigma) + 2Q(\sqrt{5}/\sigma) \\
&\qquad + Q(2\sqrt{2}/\sigma) + 2Q(\sqrt{37}/2\sigma) \\
&\approx 2Q(1/\sigma)\,. && \text{(S.245)}
\end{aligned}
$$

(c) The noise components in the directions of the nearest neighbors are not orthogonal, and hence not independent.

(d) The average power in the 16-QAM constellation is $10c^2$ and in the V.29 constellation it is 13.5, so $c = \sqrt{1.35} \approx 1.16$. The approximate probabilities of error are $4Q(1.16/\sigma)$ for 16-QAM and $2Q(1/\sigma)$ for V.29, so assuming the SNR is high enough that the constant multipliers are not important, 16-QAM is about 1.3 dB better. There are good reasons, nonetheless, for using the V.29 constellation. In particular, it is less sensitive to phase jitter.

Problem 5-21. This implies that the folded spectrum is a constant, $S_h(e^{j2\pi fT}) = E_h$, which admits a trivial spectral factorization (5.67) with $\gamma^2 = E_h$ and $M(z) = 1$. The precursor filter of (5.69) reduces to a constant gain of E_h^{-1}. In this case, the minimum-distance sequence detector of (5.72) reduces to:

$$\{ \hat{a}_k \} = \arg \min_{\{a_k\} \in \mathcal{A}^L} \sum_{k=0}^{L-1} \left| \frac{y_k}{E_h} - a_k \right|^2 . \tag{245.1}$$

This makes intuitive sense, since the output of the sampled matched filter is $y_k = E_h a_k + n_k$, where n_k is unknown noise. When the data symbols are chosen independently of one another, the minimization can be performed symbol-by-symbol: since all the terms in the sum are non-negative, the sum is minimized by minimizing each term, where each term is precisely the criterion for a minimum-distance slicer design. Thus, in this case, the simplified structure of Fig. 5-25(b) can be used. This is exactly the same structure of Fig. 5-19 that was derived for the special case of an isolated pulse, except that here the matched filter output is sampled every symbol period instead of only once at time zero. Thus, with independent symbols and no ISI, the minimum-distance sequence detector reduces to a sequence of independent one-shot minimum-distance detectors.

Problem 5-22. The allpass filter has transfer function

$$H_{\text{allpass}}(z) = \frac{d^* - z^{-1}}{1 - dz} , \qquad |d| < 1 . \tag{S.246}$$

(a) The precursor equalizer filter is

$$\frac{(1 - c^* z)(d^* - z^{-1})}{1 - dz^{-1}} \tag{S.247}$$

and the channel model filter is

$$\frac{d^* - z^{-1}}{(1 - cz^{-1})(1 - dz^{-1})} . \tag{S.248}$$

(b) The zero location of the upper-path filter is outside the unit circle, since it is maximum-phase. In order to cancel this zero, the pole of the allpass filter would have to be outside the unit circle, which would make it anti-causal.

(c) The filter transfer function can be expanded in partial fractions as

$$\frac{a}{1 - cz^{-1}} + \frac{b}{1 - dz^{-1}} \tag{S.249}$$

which has impulse response

$$ac^k u_k + bd^k u_k , \tag{S.250}$$

where

$$a = (cd^* - 1)/(c - d) , \qquad b = (|d|^2 - d^2)/(d - c) . \tag{S.251}$$

Problem 5-23. The channel is so noisy that, even with binary PAM, a bit-error probability of 10^{-6} cannot be maintained. Indeed, with the alphabet $\mathcal{A} = \{\pm 1\}$ on this channel, the bit-error probability can be found exactly:

$$P_b = Q(1/\sigma) = Q(1/\sqrt{0.1}) = 7.8 \times 10^{-4}. \tag{S.252}$$

Increasing the size of the alphabet while maintaining unit energy will only make the bit-error probability grow larger.

Note to instructor: Changing the noise standard deviation from $\sigma = \sqrt{0.1}$ to $\sigma = 0.006$ will make the problem much more interesting. A general solution for arbitary SNR is outlined below.

To proceed we need to know the error probability of L-ary PAM. We've already done this in (5.104) for the special case of 4-ary PAM, but we need to generalize to arbitrary L. The probability of error for L-ary PAM is easily derived:

$$\begin{aligned} P_e &= \Pr[a \text{ on end}]\Pr[\text{error} \,|\, a \text{ on end}] \\ &\quad + \Pr[a \text{ not on end}]\Pr[\text{error} \,|\, a \text{ not on end}] \qquad \text{(S.253)} \\ &= \frac{2}{L}\Big(Q(d/2\sigma)\Big) + \frac{L-2}{L}\Big(2Q(d/2\sigma)\Big) \\ &= 2\Big(1 - \frac{1}{L}\Big)Q(c/\sigma) \\ &= 2\Big(1 - \frac{1}{L}\Big)Q\Big(\sqrt{\frac{3SNR}{L^2-1}}\Big), \qquad \text{(S.254)} \end{aligned}$$

where we substituted $c = \sqrt{(3E_a)/(L^2-1)}$ and $SNR = \mathrm{E}[X^2]/\mathrm{E}[N^2] = E_a/\sigma^2$. (Observe that this reduces to (5.104) when $L = 4$, as expected.). Assuming a Gray mapping, the bit-error probability will be:

$$P_b = \frac{1}{\log_2 L} P_e . \tag{S.255}$$

Solving this equation for the L^2 term in the denominator of (S.254), as a function of P_b, yields:

$$L^2 = 1 + \frac{SNR}{\Gamma} , \tag{S.256}$$

where

$$\Gamma = \frac{1}{3}\Big(Q^{-1}\Big(\frac{P_b \log_2 L}{2(1-1/L)}\Big)\Big)^2 . \tag{S.257}$$

With $P_b = 10^{-6}$, the factor Γ ranges from 7.5 to 6.7 as L ranges from 2 to 128. If we approximate Γ by $\Gamma \approx 7$, independent of L, then (S.256) gives:

$$L \approx \sqrt{1 + \frac{SNR}{7}} . \tag{S.258}$$

This is how big the alphabet can be while maintaing $P_b = 10^{-6}$. If the noise standard deviation were $\sigma = 0.006$, then substituting $SNR = 1/\sigma^2$ gives $L = 63$. But at this value of L, our approximation of $\Gamma = 7$ can be improved upon, since Γ is known to be 6.82 when $L = 64$. With $\Gamma = 6.82$, (S.256) gives $L = 63.8$, or essentially $L = 64$.

Problem 5-24.

(a) The precursor equalizer filter is

$$\frac{d^* - z^{-1}}{(1 - c^* z)(1 - dz^{-1})} \tag{S.259}$$

and the channel-model filter is

$$\frac{(1 - cz^{-1})(d^* - z^{-1})}{1 - dz^{-1}} . \tag{S.260}$$

(b) The pole location of the precursor equalizer filter is outside the unit circle, since it is maximum-phase. In order to cancel this pole, the zero of the allpass filter would have to be outside the unit circle, which is acceptable because the allpass filter is then causal. The precursor equalizer filter then has transfer function

$$\frac{-z^{-1}}{1 - cz^{-1}} , \tag{S.261}$$

which is causal, and the channel-model filter transfer function is $c^* - z^{-1}$.

(c) The filter transfer function can be expanded in partial fractions as

$$\gamma^2 \frac{d^* - (cd^* + 1)z^{-1} + cz^{-2}}{1 - dz^{-1}} \tag{S.262}$$

which has impulse response

$$\gamma^2\left(d^* d^k u_k - (cd^* + 1)d^{k-1} u_{k-1} + cd^{k-2} u_{k-2}\right) . \tag{S.263}$$

(d) The impulse response is FIR,

$$c^* \delta_k - \delta_{k-1} . \tag{S.264}$$

Problem 5-25. Observe that $E[Z(t)]$ has zero mean, since $E[\cos(2\pi f_c t + \Theta)] = 0$. To find its autocorrelation function, use the trigonometric identity

$$cos(a)cos(b) = 0.5\cos(a - b) + 0.5\cos(a + b) \tag{S.265}$$

to get

$$\begin{aligned} R_Z(\tau) &= 2E[\cos(2\pi f_c(t + \tau) + \Theta)\cos(2\pi f_c t + \Theta)S(t + \tau)S^*(t)] \\ &= E[\cos(2\pi f_c \tau) + \cos(2\pi f_c(2t + \tau) + 2\Theta)]R_S(\tau) \\ &= \cos(2\pi f_c \tau)R_S(\tau) . \end{aligned} \tag{S.266}$$

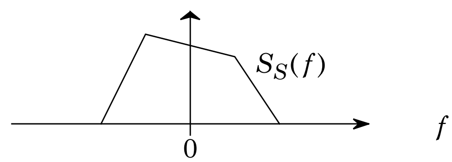
We have used the easily shown fact that

$$E[\cos(2\pi f_c(2t + \tau) + 2\Theta)] = 0. \tag{S.267}$$

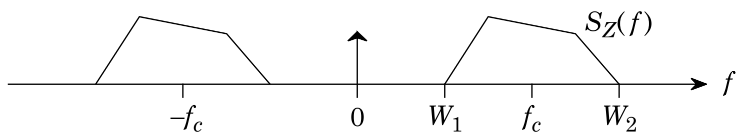
The power spectrum is then given by

$$S_Z(f) = 0.5[S_S(f_c - f) + S_S(-f_c - f)] \quad . \tag{S.268}$$

For example, given the same complex-valued baseband power spectrum shown below,



the power spectrum of $Z(t)$ is shown below:



Without the random phase, $Z(t)$ is not WSS. This is because $\mathrm{E}[Z(t)Z(t+\tau)]$ depends on t as well as τ when the phase is $\Theta = 0$. For example, when $t = 0$, we have $\mathrm{E}[Z(0)Z(0+\tau)] = 0$ for all τ. But other values of t will give a nonzero autocorrelation.

Problem 5-26. We can write

$$X(t) = \sqrt{2}\,\mathrm{Re}\{Z(t)\} = \frac{1}{\sqrt{2}}[Z(t) + Z^*(t)] \tag{S.269}$$

where

$$Z(t) = S(t)e^{j(2\pi f_c t + \Theta)} \quad . \tag{S.270}$$

Showing that the expected value is independent of time is easy using

$$E[X(t)] = \sqrt{2}\,\mathrm{Re}\{E[S(t)]E[e^{j(2\pi f_c t + \Theta)}]\} \quad . \tag{S.271}$$

The second expectation is easily shown to be zero by directly computing it (the expectation is over Θ, which has a uniform p.d.f., so we simply need to integrate over $[0, 2\pi]$). The autocorrelation can be written

$$E[X(t_1)X^*(t_2)] = \frac{1}{2}E[Z(t_1)Z(t_2) + Z(t_1)Z^*(t_2) + Z^*(t_1)Z(t_2) + Z^*(t_1)Z^*(t_2)] \quad , \tag{S.272}$$

where the first term,

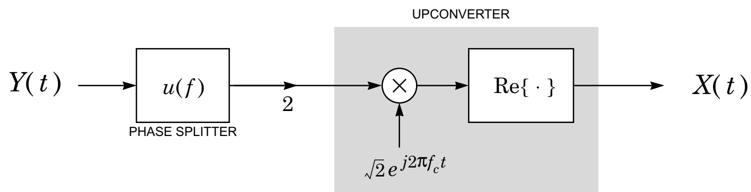
$$E[Z(t_1)Z(t_2)] = E[S(t_1)S(t_2)]E[e^{j(2\pi f_c(t_1 + t_2) + 2\Theta)}] \quad . \tag{S.273}$$

can also easily be shown to be zero by directly computing the expectation over Θ of the complex exponential. The fourth term $E[Z^*(t_1)Z^*(t_2)]$ will similarly turn out to be zero. The second term is

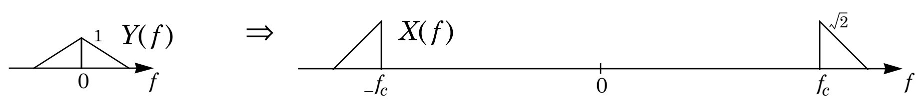
$$E[Z(t_1)Z^*(t_2)] = E[S(t_1)S^*(t_2)]e^{j(2\pi f_c(t_1 - t_2)} \quad . \tag{S.274}$$

The exponential is now a deterministic function of $t_1 - t_2$, as is $E[S(t_1)S^*(t_2)]$, the autocorrelation of $S(t)$, because $S(t)$ is WSS. The third term will similarly prove to be a function of $t_1 - t_2$, so $X(t)$ is WSS.

Problem 5-27. Before we begin, note from (2.23) that $(Y(t) + j\hat{Y}(t))/2$ is the output of a phase splitter whose input is $Y(t)$, and therefore $X(t)$ is related to $Y(t)$ by the following picture:



This system is thus a *upper-single-sideband modulator*; for example, if the input $Y(t)$ is deterministic with Fourier transform $Y(f)$, as shown below, the output Fourier transform $X(f)$ is as shown below:



We want to show that the WSS property is preserved by the U-SSB modulator. The output signal can be written as:

$$X(t) = \sqrt{2}\left(Y(t)\cos(2\pi f_c t) - \hat{Y}(t)\sin(2\pi f_c t)\right). \tag{S.275}$$

We want to show that the following autocorrelation function is independent of time t:

$$\begin{aligned}
R_{XX}(t+\tau, t) &= \mathrm{E}[X(t+\tau)X(t)] \\
&= 2\mathrm{E}\Big[\Big(Y(t+\tau)\cos(2\pi f_c(t+\tau)) - \hat{Y}(t+\tau)\sin(2\pi f_c(t+\tau))\Big) \times \\
&\qquad \Big(Y(t)\cos(2\pi f_c t) - \hat{Y}(t)\sin(2\pi f_c t)\Big)\Big] \\
&= 2\mathrm{E}\Big[Y(t+\tau)Y(t)\Big]\cos(2\pi f_c(t+\tau))\cos(2\pi f_c t) \\
&\quad - 2\mathrm{E}\Big[\hat{Y}(t+\tau)Y(t)\Big]\sin(2\pi f_c(t+\tau))\cos(2\pi f_c t) \\
&\quad - 2\mathrm{E}\Big[Y(t+\tau)\hat{Y}(t)\Big]\cos(2\pi f_c(t+\tau))\sin(2\pi f_c t) \\
&\quad + 2\mathrm{E}\Big[\hat{Y}(t+\tau)\hat{Y}(t)\Big]\sin(2\pi f_c(t+\tau))\sin(2\pi f_c t) \qquad \text{(S.276)} \\
&= 2R_Y(\tau)\cos(2\pi f_c(t+\tau))\cos(2\pi f_c t) \\
&\quad - 2R_{\hat{Y}Y}(\tau)\sin(2\pi f_c(t+\tau))\cos(2\pi f_c t)
\end{aligned}$$

$$-2R_{Y\hat{Y}}(\tau)\cos(2\pi f_c(t+\tau))\sin(2\pi f_c t)$$
$$+2R_{\hat{Y}}(\tau)\sin(2\pi f_c(t+\tau))\sin(2\pi f_c t). \tag{S.277}$$

Since the Hilbert transformer is an all-pass filter, $S_{\hat{Y}}(f) = S_Y(f)$, and hence $R_{\hat{Y}}(\tau) = R_Y(\tau)$. Furthermore, if $h(t)$ is the impulse response of the Hilbert transformer, then:

$$\begin{aligned}\mathrm{E}[\hat{Y}(t+\tau)Y(t)] &= \mathrm{E}\Big[\int_{-\infty}^{\infty} h(u)Y(t+\tau-u)du\,Y(t)\Big] \\ &= \int_{-\infty}^{\infty} h(u)\mathrm{E}\Big[Y(t+\tau-u)Y(t)\Big]du \\ &= \int_{-\infty}^{\infty} h(u)R_Y(\tau-u)du = R_Y(\tau) * h(\tau),\end{aligned} \tag{S.278}$$

which implies that:

$$R_{\hat{Y}Y}(\tau) = \hat{R}_Y(\tau) \tag{S.279}$$

and

$$S_{\hat{Y}Y}(f) = -j\mathrm{sign}(f)S_Y(f). \tag{S.280}$$

In particular, since $S_{\hat{Y}Y}(f)$ is purely imaginary, $R_{\hat{Y}Y}(\tau)$ must be odd, satisfying $R_{\hat{Y}Y}(-\tau) = -R_{\hat{Y}Y}(\tau)$. And $R_{\hat{Y}Y}(\tau)$ is related to $R_{Y\hat{Y}}(\tau)$ by:

$$\begin{aligned}R_{\hat{Y}Y}(\tau) &= \mathrm{E}[\hat{Y}(t'-\tau+\tau)Y(t'-\tau)] \quad (\text{when } t = t'-\tau) \\ &= \mathrm{E}[\hat{Y}(t')Y(t'-\tau)] \\ &= R_{Y\hat{Y}}(-\tau).\end{aligned} \tag{S.281}$$

Therefore, (S.277) reduces to:

$$R_{XX}(t+\tau, t) =$$
$$2R_Y(\tau)\Big(\cos(2\pi f_c(t+\tau))\cos(2\pi f_c t) + \sin(2\pi f_c(t+\tau))\sin(2\pi f_c t)\Big)$$
$$-2R_{\hat{Y}Y}(\tau)\Big(\sin(2\pi f_c(t+\tau))\cos(2\pi f_c t) - \cos(2\pi f_c(t+\tau))\sin(2\pi f_c t)\Big) \tag{S.282}$$

$$= 2R_Y(\tau)\cos(2\pi f_c\tau) - 2R_{\hat{Y}Y}(\tau)\sin(2\pi f_c\tau), \tag{S.283}$$

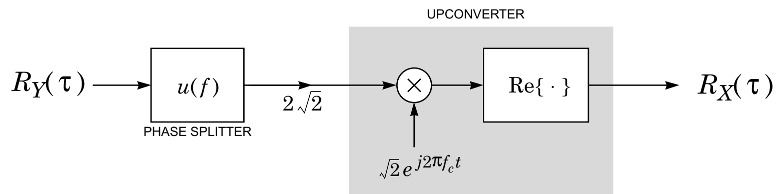
where we used the trigonemetric identities:

$$\begin{aligned}\cos(x)\cos(y) + \sin(x)\sin(y) &= \cos(x-y)\,, \\ \sin(x)\cos(y) - \cos(x)\sin(y) &= \sin(x-y)\,.\end{aligned} \tag{S.284}$$

Clearly (S.283) is independent of time t, so that $X(t)$ is indeed WSS. Further, since $R_{\hat{Y}Y}(\tau) = \hat{R}_Y(\tau)$, we recognize (S.283) as being the output of an upconverter:

$$R_X(\tau) = 2\mathrm{Re}\{\Big(R_Y(\tau) + j\hat{R}_Y(\tau)\Big)e^{j2\pi f_c t}\}. \tag{S.285}$$

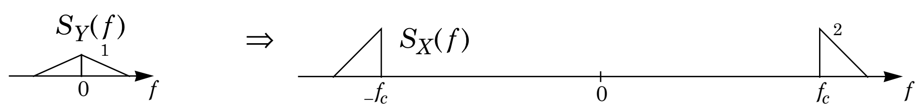
In particular, this is the output of an upconverter whose input is $\sqrt{2}\left(R_Y(\tau) + j\hat{R}_Y(\tau)\right)$, which from (2.23) is a scaled version of the output of a phase splitter whose input is $R_Y(\tau)$. Overall, the relationship between $R_Y(\tau)$ and $R_X(\tau)$ is captured by the following picture:



Except for the the factor of $\sqrt{2}$, this is identical to the U-SSB modulator that we began with. Therefore, the power spectrum is:

$$S_X(f) = 2S_Y(f - f_c)u(f - f_c) + 2S_Y(f + f_c)u(f - f_c). \tag{S.286}$$

An example is sketched below:



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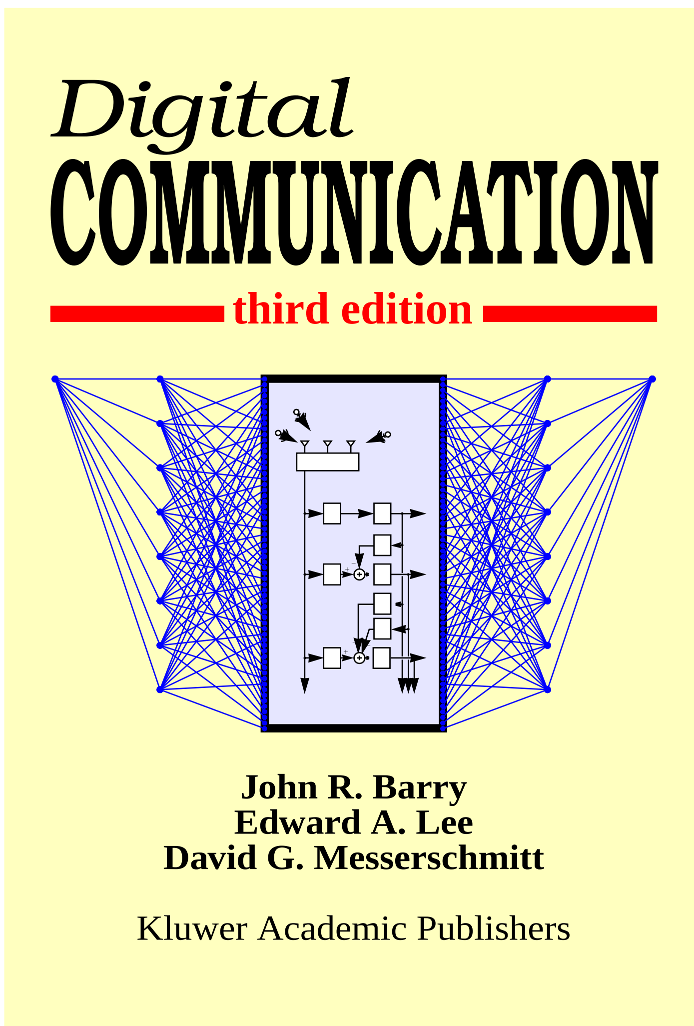


Chapter 6

Problem 6-1.

$$\int_{-\infty}^{\infty} h_0(t)h_1(t)dt = \frac{2E}{T}\int_0^T \sin(2\pi f_0 t)\sin(2\pi f_1 t)dt$$
$$= \frac{E}{T}\int_0^T \Big(\cos(2\pi(f_0 - f_1)t) + \cos(2\pi(f_0 + f_1)t)\Big)dt. \tag{S.287}$$

Under both set of assumptions, the integral evaluates to zero.

Problem 6-2. Begin by rewriting the pulses $g_i(t) = \pm\sin(2\pi f_i t)w(t)$ from (6.114) as:

$$g_0(t) = \pm\ \sin(2\pi(f_c + f_d)t)w(t)$$
$$g_1(t) = \pm\ \sin(2\pi(f_c - f_d)t)w(t) \tag{S.288}$$

where

$$f_c = \frac{f_0 + f_1}{2}\ , \qquad f_d = \frac{f_0 - f_1}{2} \tag{S.289}$$

are the nominal carrier frequency and the peak deviation frequency, respectively. Then

$$\int_0^T g_0(t)g_1(t)dt = \pm\frac{1}{2}\int_0^T \cos(4\pi f_d t)dt \ \pm\ \frac{1}{2}\int_0^T \cos(4\pi f_c t)dt$$
$$= \frac{1}{8\pi f_d}\sin(4\pi f_d T)\ . \tag{S.290}$$

The signals are orthogonal if and only if this is zero, which occurs if and only if

$$4\pi f_d T = K\pi \tag{S.291}$$

for some integer K. The minimum (non-zero) frequency spacing therefore occurs when

$$4\pi f_d T = \pi \tag{S.292}$$

or

$$f_d = \frac{1}{4T}\ . \tag{S.293}$$

This is the frequency spacing (6.113) of MSK signals.

Problem 6-3. For MSK, the frequency separation should be $f_i - f_{i-1} = 1/(2T) = 0.5$ MHz. So the desired frequencies are $f_1 = 10.5$ MHz, $f_2 = 11.0$ MHz, etc.

Problem 6-4.

(a) From (6.117), using trigonometric identities,

$$x(t) = \sum_{m=-\infty}^{\infty}\Big(\sin(2\pi f_c t + \frac{b_k\pi t}{2T})\cos(\phi_k) + \cos(2\pi f_c t + \frac{b_k\pi t}{2T})\sin(\phi_k)\Big)w(t-kT) \quad . \tag{S.294}$$

Since $\sin(\phi_k) = 0$ this becomes, using more trigonometric identities

$$x(t) = \sum_{m=-\infty}^{\infty}\Big(\sin(2\pi f_c t)\cos\Big(\frac{b_k\pi t}{2T}\Big) + \cos(2\pi f_c t)\sin\Big(\frac{b_k\pi t}{2T}\Big)\Big)\cos(\phi_k)w(t-kT) \, . \tag{S.295}$$

From the symmetry of the cosine we get

$$\cos\Big(\frac{b_k\pi t}{2T}\Big) = \cos\Big(\frac{\pi t}{2T}\Big) \tag{S.296}$$

and from the anti-symmetry of the sine we get

$$\sin\Big(\frac{b_k\pi t}{2T}\Big) = b_k\sin\Big(\frac{\pi t}{2T}\Big) \; , \tag{S.297}$$

from which the result follows.

(b) Notice that $b_{k-1} - b_k$ is always either zero or ± 2, so if k is even then from (6.118)

$$\phi_k = \phi_{k-1} + K2\pi \; , \tag{S.298}$$

where K is an integer. Hence

$$Q_k = \cos(\phi_k) = Q_{k-1} = \cos(\phi_{k-1}) \quad . \tag{S.299}$$

If k is odd then examining (6.118) we see that if $b_k = b_{k-1}$ then $I_k = I_{k-1}$. Furthermore, if $b_k = -b_{k-1}$ then

$$\phi_k = \phi_{k-1} \pm \pi k \tag{S.300}$$

and since k is odd, $\cos(\phi_k) = -\cos(\phi_{k-1})$, and again $I_k = I_{k-1}$.

(c) Write the first summation of (6.179)

$$\cos(2\pi f_c t)\Sigma_{k \text{ even}} I_k\sin\Big(\frac{\pi t}{2T}\Big)[w(t-kT) - w(t-kT-T)] \tag{S.301}$$

using the fact that for k even $I_k = I_{k+1}$. Then notice that for k even

$$\sin\Big(\frac{\pi(t-kT)}{2T}\Big) = (-1)^{k/2}\sin\Big(\frac{\pi t}{2T}\Big) \tag{S.302}$$

so

$$\sin\left(\frac{\pi t}{2T}\right) = \sin\left(\frac{\pi(t-kT)}{2T}\right)(-1)^{k/2} \ . \tag{S.303}$$

Hence the first summation of (6.179) is

$$\cos(2\pi f_c t)\sum_{k \text{ even}} I_k(-1)^{k/2}p(t-kT) \ . \tag{S.304}$$

The second summation follows similarly.

Problem 6-5. From(2.23), the output of the phase splitter when the input is $A\cos(2\pi f_c t)$ is:

$$\begin{aligned}\frac{1}{2}\left(s(t) + j\hat{s}(t)\right) &= \frac{A}{2}\left(\cos(2\pi f_c t) + j\cos(2\pi f_c t)\right) \\ &= \frac{A}{2}e^{j2\pi f_c t} \ .\end{aligned} \tag{S.305}$$

Clearly, the magnitude of this complex signal is $A/2$.

Problem 6-6. Each of 128 subchannels (one per pulse) must carry a bit rate of $19{,}200/128 = 150$ bits per second. Using 4-PSK, we can transmit 2 bits per symbol, so the symbol rate should be 75 symbols/second. Hence, T is $1/75$ seconds or about 13 msec.

Problem 6-7. The sketches are shown in Fig. S-3. Define

$$s_n(t) = \frac{1}{2\sqrt{T}}\left(\frac{\sin(\pi t/(2T))}{\pi t/(2T)}\right)\cos((n+1/2)\pi t/T) \ . \tag{S.306}$$

Then

$$h_0(t) = s_1(t) + s_3(t) \ , \tag{S.307}$$

$$h_1(t) = s_1(t) - s_3(t) \ , \tag{S.308}$$

$$h_2(t) = s_2(t) + s_0(t) \ , \tag{S.309}$$

$$h_3(t) = s_2(t) - s_0(t) \ . \tag{S.310}$$

The bandwidth efficiency is the same as that of the pulses in Fig. 6-8.

Problem 6-8.

(a) This is just a matter of multiplying it out.

(b) Let $\mathbf{H}(f) = \sqrt{T/3}\,\mathbf{D}(f)$.

(c) If we choose $M_1 = 0$ and $M_2 = 2$ in (6.166), we get the pulses with Fourier transforms shown in Fig. S-4. From this, we see that each pulse consists of three parts. Each part is a sinc pulse modulated by $e^{-j2\pi mt/T}$ for $m = 0, 1,$ and 2, and scaled by a complex value that depends on n. For $n = 0, 1,$ and 2, we can write

$$h_n(t) = \frac{1}{\sqrt{3T}} \frac{\sin(\pi t/T)}{\pi t/T}\left(1 + e^{-j2\pi(t/T+n/3)} + e^{-j4\pi(t/T+n/3)}\right). \tag{S.311}$$

(d) The time domain pulses are not real. To use these pulses for orthogonal multipulse over a real channel, we need to upconvert them, forming the real-valued passband equivalent pulses

$$h'_n(t) = \sqrt{2}\,\mathrm{Re}\{e^{j2\pi f_c t} h_n(t)\} \tag{S.312}$$

for some carrier $f_c \geq 1/(2T)$. The upconversion doubles the bandwidth to N/T Hz, making the spectral efficiency of orthogonal multipulse

$$\nu = \frac{\log_2 N}{N} = \frac{\log_2 3}{3} \approx .528\ . \tag{S.313}$$

With combined PAM and multipulse, instead of transmitting $\log_2(3)$ bits per symbol we can transmit $\log_2(M)$ on each of N simultaneously transmitted pulses, so

$$\nu = \frac{N \log_2 M}{N} = \log_2(M)\ . \tag{S.314}$$

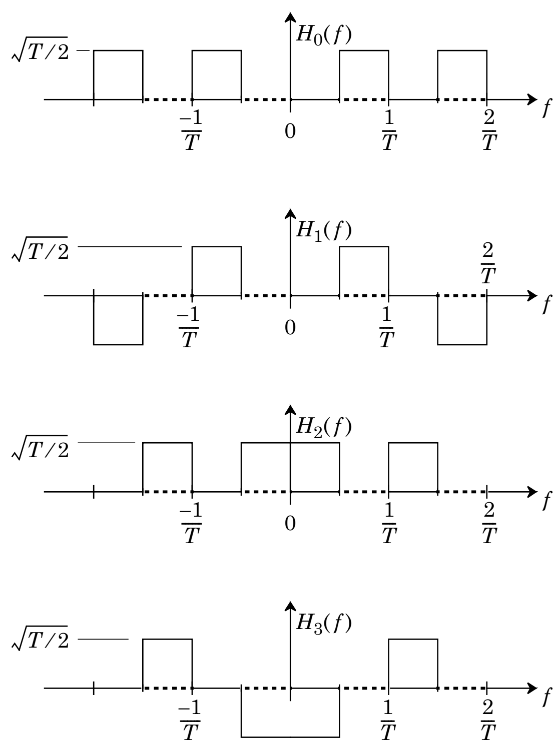


Fig. S-3. The Fourier transform for the orthonormal pulses in Problem 6-7. The value of $H_n(f)$ in the dashed regions of the f axis is determined by the requirement for conjugate symmetry, while the value in the solid regions is determined by $\mathbf{H}(f)$.

This is equivalent to passband PAM with the same alphabet size, and since symbols can be complex, is equivalent to baseband PAM and to PAM plus multipulse. The pulses are not practical because of the gradual rolloff of the sinc function, or equivalently, because of the abrupt transitions in the frequency domain.

Problem 6-9. The conditions of the problem are satisfied if

$$\| \mathbf{Y} - \mathbf{S}_j \|^2 \le \| \mathbf{Y} - \mathbf{S}_i \|^2 \tag{S.315}$$

and substituting for **Y**, this becomes

$$\|\mathbf{E} + \mathbf{S}_i - \mathbf{S}_j \|^2 \le \|\mathbf{E}\|^2 . \tag{S.316}$$

Multiplying out the left side, we get

$$\|\mathbf{E}\|^2 + d_{i,j}{}^2 - 2\mathrm{Re}\{\langle \mathbf{E}, \mathbf{S}_j - \mathbf{S}_i \rangle\} \le \|\mathbf{E}\|^2 , \tag{S.317}$$

and dividing by $d_{i,j}$,

$$\mathrm{Re}\{ \langle \mathbf{E}, \frac{\mathbf{S}_j - \mathbf{S}_i}{d_{i,j}} \rangle\} \ge d_{i,j} / 2 . \tag{S.318}$$

This is the result promised, since $(\mathbf{S}_j - \mathbf{S}_i) / d_{i,j}$ is a unit vector in the direction of $(\mathbf{S}_j - \mathbf{S}_i)$.

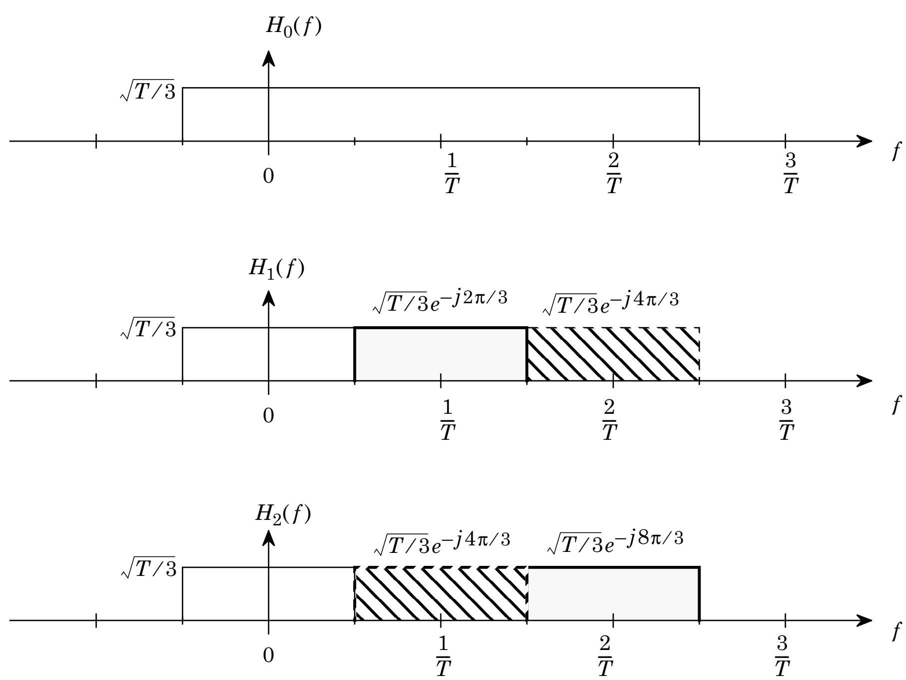


Fig. S-4. The Fourier transforms of three orthogonal multipulse pulses. Complex values are plotted as magnitudes but labeled with the actual complex value.

Problem 6-10.

(a) If the data symbols have magnitude unity, then the minimum distance between them is $d_{min} = \sqrt{2}$.

(b) d_{min} is σ_h times the minimum distance between data symbols, and hence $\sqrt{2}\,\sigma_h$.

$$d^2_{\min} = \sigma_h{}^2 + \sigma_h{}^2 \quad \Rightarrow \quad d_{\min} = \sqrt{2}\,\sigma_h \,. \tag{S.319}$$

(c) The distance is

$$d^2 = |\varepsilon_1|^2 + |\varepsilon_2|^2 + 2\alpha\mathrm{Re}\{\varepsilon_1\varepsilon_2{}^*\} \,. \tag{S.320}$$

Clearly we want to make the third term as large negative as we can. Since the ε can be have a phase that is any multiple of $\pi/4$, clearly the third term is minimized if ε_1 and ε_2 are antipodal; that is, they have opposite phase. Also, the ε can have magnitude either $\sqrt{2}$ or 2. Thus, there are three cases to consider:

$$d^2_{\min} = 4 + 4 - 2\alpha \cdot 2 \cdot 2 \tag{S.321}$$

$$d^2_{\min} = 4 + 2 - 2\alpha \cdot \sqrt{2} \cdot 2 \tag{S.322}$$

$$d^2_{\min} = 2 + 2 - 2\alpha \cdot \sqrt{2} \cdot \sqrt{2} \,. \tag{S.323}$$

It is easy to verify that the third case is always the smallest for all $0 \le \alpha < 1$, and hence

$$d_{\min} = 2\sqrt{1-\alpha} \,. \tag{S.324}$$

Note that the minimum distance goes to zero as $\alpha \rightarrow 1$.

Problem 6-11. Let us bound the energy of $s(t)$ in the interval $[KT, \infty)$,

$$\begin{aligned}\int_{KT}^{\infty} s^2(t)\,dt &= \int_{KT}^{\infty}\Big(\sum_{k=0}^{K-1}\sum_{n=1}^{N} a_{k,n}h_n(t-kT)\Big)^2 dt \\ &\le \sum_{k=0}^{K-1}\sum_{n=1}^{N} a_{k,n}{}^2\int_{KT}^{\infty} h_n{}^2(t-kT)\,dt \\ &\le \sum_{k=0}^{K-1}\sum_{n=1}^{N} a_{k,n}{}^2\frac{\alpha}{((k-K)T)^2}\,.\end{aligned} \tag{S.325}$$

If the data symbols are drawn from a finite constellation, we can assume that $\sum_{n=1}^{N} a_{k,n}{}^2$ is bounded, say by C_1. The remaining sum:

$$\sum_{k=0}^{K-1} 1/(KT-kT)^2 \tag{S.326}$$

is also bounded, since the series is convergent, say by C_2. Hence, the energy outside $[0, KT]$ is bounded by a constant $\alpha C_1 C_2$, independent of K, and the fraction of the energy outside the interval $[KT, \infty]$ goes to zero since the energy of $s(t)$ is increasing with K.

Problem 6-12. From (5.52) and (5.53), the signal-to-noise ratio is defined by:

$$\text{SNR} = \frac{|p(0)|^2 E_a}{E_f N_0} \tag{S.327}$$

where $E_a = \text{E}[\,|a_k|^2]$ is the energy in the symbols, E_f is the energy in the receive filter $f(t)$, and $p(0)$ is the *overall* pulse shape (after the receive filter) evaluated at time zero. The transmit pulse shape $g(t)$ is specified as the raised-cosine pulse shape given in (5.8) with $\alpha = 1$ (100% excess bandwidth). We have already established (see the solution to Problem 5-2) that the energy of this pulse is $E_g = (1 - \alpha/4)T = 3T/4$. Therefore, the transmit signal power (average energy per unit time) will be:

$$P = \frac{E_a E_g}{T} = \frac{3}{4} E_a. \tag{S.328}$$

Therefore, to ensure that thethe transmit power is limited to $P = 1$, we must have:

$$E_a = \frac{4}{3}. \tag{S.329}$$

The receive filter is specified in the frequency domain as $F(f) = 1$ for $|f| < 1/T$ and $F(f) = 0$ for $|f| > 1/T$, which has two consequences. First, it is easy to see that its energy is:

$$E_f = \int_{-\infty}^{\infty} |F(f)|^2 df = \frac{2}{T}. \tag{S.330}$$

Second, the receive filter will not affect the pulse shape, so that the overall pulse shape is just the transmit pulse shape:

$$p(t) = g(t) \quad \Rightarrow \quad p(0) = 1. \tag{S.331}$$

The SNR is thus:

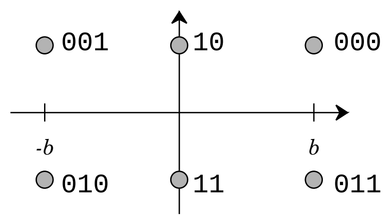
$$\text{SNR} = \frac{p^2(0) E_a}{E_f N_0} = \frac{2T}{3N_0}. \tag{S.332}$$

Therefore, from (5.113) with $M = 16$:

$$\begin{aligned} P_e &= 4\left(1 - \frac{1}{\sqrt{M}}\right) Q\left(\sqrt{\frac{3SNR}{M-1}}\right) - 4\left(1 - \frac{1}{\sqrt{M}}\right) Q^2\left(\sqrt{\frac{3SNR}{M-1}}\right) \\ &= 3Q\left(\sqrt{\frac{SNR}{5}}\right) - 2.25 Q^2\left(\sqrt{\frac{SNR}{5}}\right) \\ &= 3Q\left(\sqrt{\frac{2T}{15N_0}}\right) - 2.25 Q^2\left(\sqrt{\frac{2T}{15N_0}}\right). \end{aligned} \tag{S.333}$$

Problem 6-13.

(a) A suitable coder works as follows. Take one bit; if it is a one, transmit one of the inner points depending on the next bit. If it is a zero, transmit one of the outer points depending on the next two bits. An example bit mapping (which is a Gray code) is given in the figure to the right.



(b) Let C denote a correct decision and E a signal error. Then

$$Pr[C\,|\,\text{inner}] = (1 - 2Q(b/2\sigma))(1 - Q(b/2\sigma)) \tag{S.334}$$

which implies that

$$Pr[\text{symbol error}\,|\,\text{inner}] = 3Q(b/2\sigma) - 2Q^2(b/2\sigma)\ . \tag{S.335}$$

Also

$$Pr[C\,|\,\text{outer}] = (1 - Q(b/2\sigma))^2 \tag{S.336}$$

which implies that

$$Pr[\text{symbol error}\,|\,\text{outer}] = 2Q(b/2\sigma) - Q^2(b/2\sigma)\ . \tag{S.337}$$

Combining,

$$\begin{aligned} Pr[\text{symbol error}] &= \tfrac{1}{2}\,[3Q(b/2\sigma) - 2Q^2(b/2\sigma)] + \tfrac{1}{2}\,[2Q(b/2\sigma) - Q^2(b/2\sigma)] \\ &= \tfrac{5}{2}\,Q(b/2\sigma) - \tfrac{3}{2}\,Q^2(b/2\sigma)\ . \end{aligned} \tag{S.338}$$

(c) The power is

$$E[a^2] = \tfrac{1}{2}\left[\tfrac{b^2}{4}\right] + \tfrac{1}{2}\left[b^2 + \tfrac{b^2}{4}\right] = \tfrac{3}{4}\,b^2\ . \tag{S.339}$$

(d) The SNR is:

$$\begin{aligned} SNR &= \frac{E[|a|^2]}{2\sigma^2} \\ &= \frac{3b^2}{8\sigma^2} \\ &= \frac{3}{2}\left(\frac{b}{2\sigma}\right)^2, \end{aligned} \tag{S.340}$$

which implies that:

$$\frac{b}{2\sigma} = \sqrt{\tfrac{2}{3}SNR}\ . \tag{S.341}$$

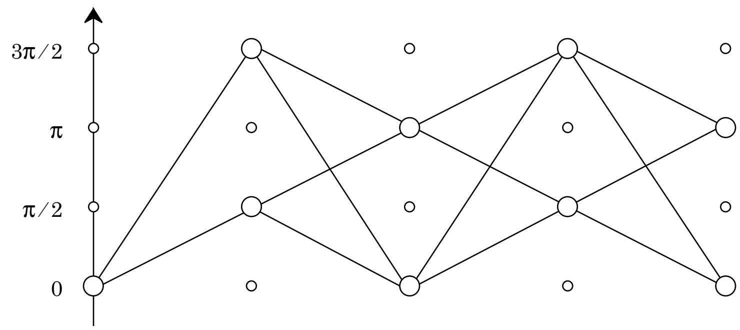


Fig. S-5. A finite planar trellis for the MSK signal.

Therefore, the answer from part (b) can be rewritten as:

$$Pr[\text{symbol error}] = \frac{5}{2} Q\left(\sqrt{\frac{2SNR}{3}}\right) - \frac{3}{2} Q^2\left(\sqrt{\frac{2SNR}{3}}\right) . \tag{S.342}$$

If SNR = 10 dB then SNR = 10 on a linear scale, so that

$$Pr[\text{symbol error}] = \frac{5}{2} Q\left(\sqrt{\frac{20}{3}}\right) - \frac{3}{2} Q^2\left(\sqrt{\frac{20}{3}}\right) = 0.0122 . \tag{S.343}$$

(e) The approximations are respectively,

$$Pr[\text{symbol error}] \approx \frac{5}{2} Q\left(\sqrt{\frac{2SNR}{3}}\right), \qquad Pr[\text{symbol error}] \approx Q\left(\sqrt{\frac{2SNR}{3}}\right) . \tag{S.344}$$

At $SNR = 10$ dB, these approximations are 0.0123 and 0.0049, respectively.

Problem 6-14.

(a) The decision regions are bounded by the planes formed by pairs of axes.

(b) The probability of error is exactly:

$$Pr[\text{symbol error}] = 1 - \left(1 - Q\left(\frac{d}{2\sigma}\right)\right)^M . \tag{S.345}$$

Problem 6-15.

(a) The trellis is shown in Fig. S-5.

(b) From (6.115), note that the MSK signal within any pulse interval is

$$g_i(t) = c\sin\left(2\pi f_c + b\frac{\pi t}{2T}\right) , \tag{S.346}$$

where $b = \pm 1$ and $c = \pm 1$. Upward-tending branches in Fig. 6-26 correspond to $b = +1$ and downward-tending branches correspond to $b = -1$. The value of c is dependent on the starting position of the branch in Fig. 6-26. The squared distance between any two branches going in opposite directions in Fig. 6-26 is

$$d^2 = \int_0^T \left[\sin\left(2\pi f_c + \frac{\pi t}{2T}\right) - \sin\left(2\pi f_c - \frac{\pi t}{2T}\right) \right]^2 dt = 2E \tag{S.347}$$

where E is the energy in one pulse. By contrast any two *distinct* (modulo 2π) *parallel* branches have squared distance $4E$. By inspection, therefore, being careful to associate branches in Fig. 6-26 with branches in Fig. S-5, we see that the minimum-distance error event is the one of length two, with squared distance $4E$.

(c) The squared distance of $4E$ is 3 dB better than the squared distance $2E$ in Fig. 6-25.

Problem 6-16. Taking a second pulse of the form of (6.100),

$$g(t) = \sum_{m=0}^{N-1} y_k h_c(t - mT_c) \ , \tag{S.348}$$

then the inner product of $h(t)$ and $g(t)$ is

$$\int_{-\infty}^{\infty} h(t) g^*(t) dt = E_c \sum_{m=0}^{N-1} x_m \, y_m^* \quad . \tag{S.349}$$

Considering $\{x_m, 0 \le m \le N-1\}$ and $\{x_m, 0 \le m \le N-1\}$ as vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ in N-dimensional Euclidean space, then the pulses $h(t)$ and $g(t)$ will be orthogonal when the Euclidean vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ are orthogonal. The number of pulses specified in this fashion that can be mutually orthogonal is $N = 2BT$, the dimensionality of the Euclidean space.

Problem 6-17. For every zero in $H(z)$ at location $z = c$, if $H(z)$ is to be allpass, we must have a pole at location $z = 1/c^*$, as shown in section 2.5.4. Since FIR filters can only have poles at $z = 0$ or $z = \infty$, an allpass FIR can only have zeros at $z = \infty$ or $z = 0$. A filter with $L \ge 0$ poles at $z = 0$ and L zeros at $z = \infty$ has impulse response δ_{k-L}. If $L < 0$, then the poles are at $z = \infty$ and the zeros are at $z = 0$.

Problem 6-18. The received signal is $r(t) = \pm h(t) + x(t)$, and the matched filter output is a random variable

$$U = \int_0^T r(t) h(t) dt = \sum_{i=1}^{N} (s_i^2 + s_i x_i) \ . \tag{S.350}$$

(a) The signal component of U is $\sum_i^N s_i^2$ and the noise component is $\sum_i^N s_i x_i$.

(b) The mean-value of the signal is E_h. The variance of the noise conditioned on knowledge of the signal is, since the x_i are independent,

$$\sum_i^N s_i^2 \mathrm{Var}[x_i] \quad . \tag{S.351}$$

The variance of the noise is the expected value of this conditional variance, which is

$$\sum_i^N E[s_i^2]\text{Var}[\,x_i\,] = \frac{E_h}{N}\sum_i^N \text{Var}[\,x_i\,] = \frac{E_h E_J}{N} \quad . \tag{S.352}$$

The SNR is therefore

$$SNR = \frac{E_h^2}{E_h E_J / N} = N\frac{E_h}{E_J} \quad . \tag{S.353}$$

The processing gain is therefore N, independent of how the jammer distributes its energy.

Problem 6-19.

(a) The group velocity is approximately $\frac{1}{2}3 \times 10^6$ meters/sec, so $\tau = 6.67 \times 10^{-7}$ seconds for a one meter link. At 10 Gb/s, the number of bits in transit is thus the product of the bit rate and τ, or 6670 bits.

(b) The transmit time is:

$$(8 \times 10^6 \text{ bits})/(10^{10} \text{ bits per second}) = 8 \times 10^{-4} \text{ seconds.} \tag{S.354}$$

A L-meter link has transit time $L \times 6.67 \times 10^{-7}$ seconds. Setting these equal, we get L = 1.2 km.

Problem 6-20. For the passband channel of Fig. 6-28-b the complex-baseband channel has bandwidth $W/2$, and hence the dimensionality of the signal subspace in time T is WT. The received vector $\mathbf{C}$ is now complex-valued, and the noise vector is complex-valued where the components are independent, have independent real and imaginary parts, each with variance $\sigma^2 = N_0/2$ (by circular symmetry of the noise). We can think of $\mathbf{C}$ as an $N = 2WT$ dimensional real-valued vector with independent noise components. The signal power constraint now applies directly to the resulting $2WT$-dimensional real-valued signal vector

$$E[\sum_i^N |s_n|^2] = E\Big[\sum_i^N \text{Re}\{s_n\}^2 + \text{Im}\{s_n\}^2\Big] = P_S T. \tag{S.355}$$

This establishes the equivalence of the baseband and passband cases.

Problem 6-21. From (6.144),

$$W_1 \log_2(1 + SNR_1) = W_2 \log_2(1 + SNR_2) \,, \tag{S.356}$$

or at high SNR,

$$SNR_1{}^{W_1} \approx SNR_2{}^{W_2} \,. \tag{S.357}$$

Taking, the logarithm, we can express the SNRs in dB, as

$$10 \cdot \log_{10} SNR_1 \approx \frac{W_2}{W_1} 10 \cdot \log_{10} SNR_2 \,. \tag{S.358}$$

Thus, for high SNR, to get the same channel capacity in half the bandwidth, the SNR (in dB) must be doubled, meaning that the SNR (not in dB) must be squared.

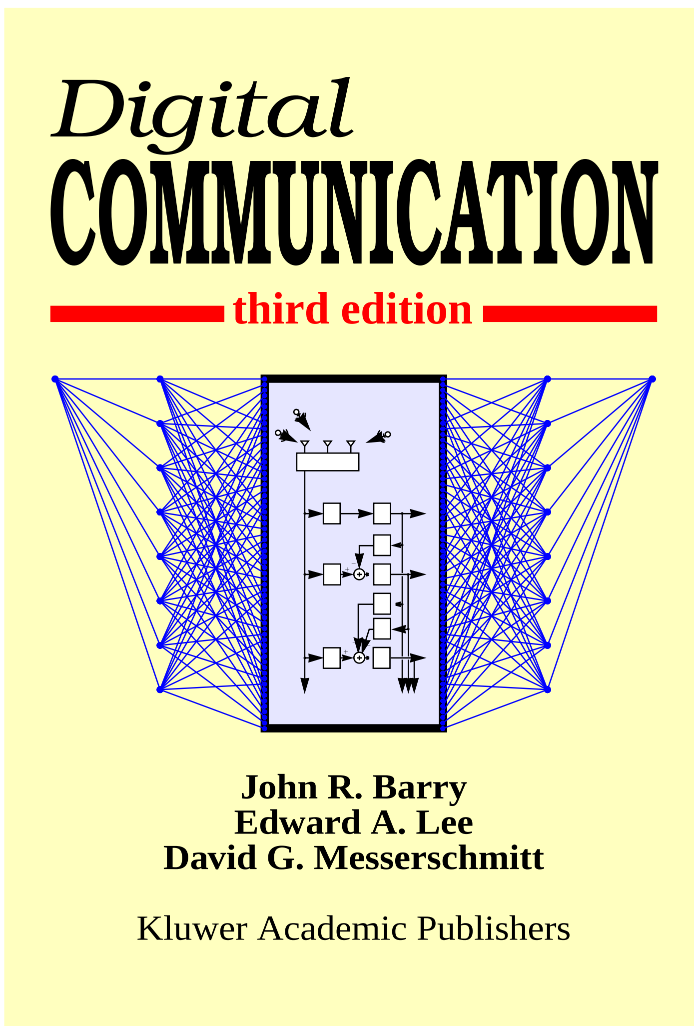
Problem 6-22. We have $M = 2$, and assuming the signal constellation is ± 1, $E_a = 1$ and $a_{min} = 2$, and $\gamma_A = 1$. The spectral efficiency is $\nu = (\log_2 2)/WT = 1/WT$, and hence

$$\gamma_{SS} = \frac{2WT(2^{1/WT} - 1)}{3} = \frac{2}{3}WT(2^{1/WT} - 1)\ . \tag{S.359}$$

Asymptotically, as $WT \to \infty$, $\gamma_{SS} \to \frac{2}{3}\log_e 2 = 0.46$. Thus, the SNR gap to capacity is asymptotically increased by $-10 \cdot \log_{10} 0.46 = 3.35$ dB.

Solutions Manual

for



Chapter 7

Problem 7-1.

(a) The ML detector ignores the a priori distribution of the input. Since $p < \frac{1}{2}$, the ML detector selects $\hat{a} = 0$ if $y = 0$ and $\hat{a} = 1$ if $y = 1$. That is, it chooses $\hat{a} = y$.

(b) An error occurs every time the channel flips a bit, which occurs with probability p.

(c) The posterior probabilities are

$$p_{Y|A}(0 \mid \hat{a}) p_A(\hat{a}) = \begin{cases} (1-p)q; & \hat{a} = 0 \\ p(1-q); & \hat{a} = 1 \end{cases}$$

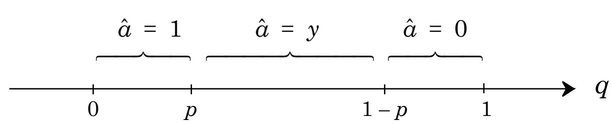
$$p_{Y|A}(1 \mid \hat{a}) p_A(\hat{a}) = \begin{cases} (1-p)q; & \hat{a} = 0 \\ p(1-q); & \hat{a} = 1 \end{cases} . \tag{S.360}$$

Using the numbers supplied we see that the MAP detector always selects $\hat{a} = 0$. An error occurs whenever $a = 1$ is transmitted, which occurs with probability $1 - q = 0.1$. This is lower than the probability of error in part (b), which is $p = 0.2$.

(d) The MAP detector will maximize the probability $p_{Y|A}(y \mid a) p_A(a)$, which is given in the following table:

a	y	$p_{Y\|A}(y \mid a) p_A(a)$
1	1	$(1-p)(1-q)$
1	0	$p(1-q)$
0	1	pq
0	0	$(1-p)q$

If we observe $y = 0$, then we will choose $\hat{a} = 0$ if $(1-p)q > p(1-q)$, or $q > p$. If we observe $y = 1$ we will choose $\hat{a} = 1$ if $(1-p)(1-q) > pq$, or $q < 1 - p$. Hence we must divide the q axis into three regions as shown below:



For very small q, the prior probability of $a = 0$ is small, so the MAP detector always chooses $\hat{a} = 1$. Similarly, for large q it always chooses $\hat{a} = 0$. In the mid-range of q, the MAP detector makes the same decision as the ML detector.

(e) The MAP detector will always select $\hat{a} = 0$ if $q > 1 - p$.

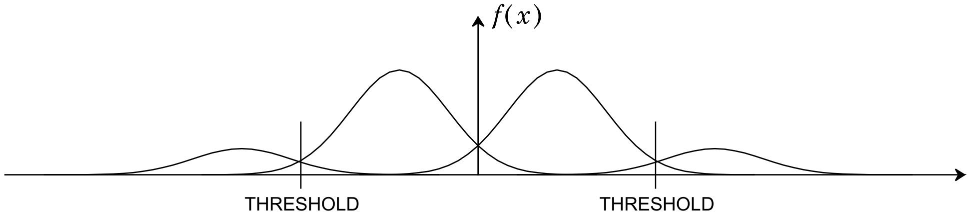
Problem 7-2. The MAP detector maximizes

$$p_{\mathbf{Y}|\boldsymbol{s}}(\boldsymbol{y}\,|\,\boldsymbol{s})p_{\boldsymbol{s}}(\boldsymbol{s}) = (1-p)^{M-W(\boldsymbol{s},\,\boldsymbol{y})}p^{W(\boldsymbol{s},\,\boldsymbol{y})}p_{\boldsymbol{s}}(\boldsymbol{s}) \ . \tag{S.361}$$

Taking the logarithm of this expression and discarding the constant term, the MAP detector equivalently minimizes

$$W(\boldsymbol{s},\boldsymbol{y})\log\left(\frac{1-p}{p}\right) - \log p_{\boldsymbol{s}}(\boldsymbol{s}) \ . \tag{S.362}$$

Problem 7-3. The values for $f_Y(y\,|\,\hat{x})p_X(\hat{x})$ are shown below for each possible $\hat{x}$ as a function of the observation *y*.



The decision regions are determined by three thresholds, the middle of which is zero. The other two can be found by finding the observation at which the receiver is indifferent between a decision +3 and a decision +1. That is the point *y* satisfying

$$|y-3|^2 - 2\sigma^2\ln(0.1) = |y-1|^2 - 2\sigma^2\ln(0.4) \ . \tag{S.363}$$

Solving this yields

$$y = 2 + \tfrac{1}{2}\sigma^2\ln(4) \ . \tag{S.364}$$

The thresholds are therefore at $\pm\,(2+\frac{1}{2}\sigma^2\ln(4))$. If $\sigma^2 = 0.25$ the thresholds are at ± 2.17, so an observation of 2.1 yields a decision $\hat{x} = 1$.

Problem 7-4. Using techniques similar to those in Problem 7-3, the final answer is

$$\frac{x_1+x_2}{2} + \frac{\sigma^2}{x_1-x_2}\ln\left(\frac{p_X(x_2)}{p_X(x_1)}\right) \tag{S.365}$$

Problem 7-5.

(a)

$$\boldsymbol{s}_1 = [1,0,0],\ \ \boldsymbol{s}_2 = [0,1,0],\ \ \boldsymbol{s}_3 = [0,0,1] \ . \tag{S.366}$$

(b)

$$\boldsymbol{s}_1 = [1,0,0],\ \ \boldsymbol{s}_2 = [0,1,0],\ \ \boldsymbol{s}_3 = [0,0,1] \ . \tag{S.367}$$

(c)

$$\Pr[\text{error}\,|\,\boldsymbol{s}_i\text{ transmitted}] \le 3Q(\sqrt{2}/2\sigma) \tag{S.368}$$

$$\Pr[\text{error} \,|\, \boldsymbol{s}_i \text{ transmitted}] \le 3Q(2, p) \quad . \tag{S.369}$$

Note that in both cases $\Pr[\text{error} \,|\, \boldsymbol{s}_i \text{ transmitted}]$ is independent of i, so

$$\Pr[\text{error}] \le 3Q(\sqrt{2}/2\sigma) \tag{S.370}$$

$$\Pr[\text{error}] \le 3Q(2, p) \quad . \tag{S.371}$$

Problem 7-6. The likelihood to be maximized is

$$\begin{aligned} L(\sigma^2) &= f_{X_1, \ldots X_N | \sigma^2}(x_1, \ldots x_N | \sigma^2) \\ &= \left(\frac{1}{2\pi\sigma^2}\right)^{N/2} e^{-x_1^2/2\sigma^2} e^{-x_2^2/2\sigma^2} \cdots e^{-x_N^2/2\sigma^2}, \end{aligned} \tag{S.372}$$

from independence. This can be rewritten as:

$$L(\sigma^2) = \left(\frac{1}{2\pi\sigma^2}\right)^{N/2} \left(\frac{-1}{2\sigma^2} \sum_{i=1}^{N} x_i^2\right)$$

Taking the derivative with respect to σ^2 we get

$$\begin{aligned} \frac{\partial}{\partial\sigma^2} L(\sigma^2) &= L(\sigma^2) \frac{\partial}{\partial\sigma^2}\left(\frac{-1}{2\sigma^2}\sum_{i=1}^{N} x_i^2\right) + \exp\left(\frac{-1}{2\sigma^2}\sum_{i=1}^{N} x_i^2\right) \frac{\partial}{\partial\sigma^2}\left(\frac{1}{2\pi\sigma^2}\right)^{N/2} \\ &= L(\sigma^2)\left(\frac{1}{2\sigma^4}\sum_{i=1}^{N} x_i^2\right) + \exp\left(\frac{-1}{2\sigma^2}\sum_{i=1}^{N} x_i^2\right)\left(\frac{1}{2\pi\sigma^2}\right)^{N/2}\left(\frac{-N}{2\sigma^2}\right) \\ &= L(\sigma^2)\left[\frac{1}{2\sigma^4}\sum_{i=1}^{N} x_i^2 - \frac{N}{2\sigma^2}\right]. \end{aligned} \tag{S.373}$$

This is zero only when the term in square brackets is zero, or when:

$$\sigma^2 = \frac{1}{N}\sum_{i=1}^{N} x_i^2 \,. \tag{S.374}$$

Thus, the ML estimator for the variance is the average of the squares of the observations.

Problem 7-7. By symmetry, we can condition the probability of error on any of the signals transmitted, and the result will be the same as the un-conditioned error probability. Hence assume that (–1, –1) is transmitted, in which case the error probability is

$$P_e = \Pr[N_1 > 1] + \Pr[N_2 > 1] - \Pr[N_1 > 1, N_2 > 1] = 2Q\left(\frac{1}{\sigma}\right) - Q^2\left(\frac{1}{\sigma}\right) \,. \tag{S.375}$$

Problem 7-8.

(a) By symmetry, it is clear that the error probability is the same whether `(000000)` or `(111111)` is transmitted, and similarly for `(111000)` or `(000111)`. If `(000000)` is transmitted, a detection error occurs whenever two out of the first three bits are in error, or two out of the last three bits, or both. If there is one channel error, no detection error is ever made, and this occurs with probability $(1-p)^6$. If there are two channel errors, if one is in the first three bits and the other is in the second three bits, no error is

made, and this occurs with probability $(3p(1-p)^2)^2 = 9p^2(1-p)^4$. However, if both errors are in the first three or second three bits, there is an error. If there are three channel errors, there must always be two errors in either the first three or the second three bits, and there is therefore always a detection error. Similarly, four or more channel errors will always result in a detection error. Thus,

$$P_e = 1 - [(1-p)^6 + 6p(1-p)^5 + 9p^2(1-p)^4] \ . \tag{S.376}$$

It is easy tp verify that the error probability is the same when `(111000)` is transmitted.

(b) Since the minimum Hamming distance is three, the approximate error probability is

$$P_e \approx 2Q(1, p) = 2[3^2(1-p) + p^3] \ . \tag{S.377}$$

When $p = 0.1$, the error probability evaluates to $P_e = 0.05522$ and the approximation is 0.056.

Problem 7-9. In AWGN, the ML detector is the minimum-distance detector. We have already solved the one-shot minimum distance detection problem for continuous-time PAM; this problem (and the next one too) requires a generalization to discrete time. A general solution, for any discrete-time impulse response and any alphabet, is easily found by following the derivation of section 5.3.1. In particular, the one-shot minimum-distance detector minimizes the following cost function:

$$\begin{aligned} J &= \sum_{k=-\infty}^{\infty} (r_k - ah_k)^2 \\ &= E_r - 2ay + a^2 E_h \,, \end{aligned} \tag{S.378}$$

where we have introduced:

$$E_r = \sum_{k=-\infty}^{\infty} r_k^2 \,, \qquad E_h = \sum_{k=-\infty}^{\infty} h_k^2 \,, \qquad y = \sum_{k=-\infty}^{\infty} r_k h_k \,. \tag{S.379}$$

Completing the square yields:

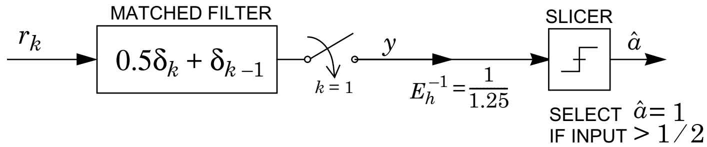
$$J = E_h \left(\frac{y}{E_h} - a \right)^2 - \frac{y^2}{E_h} + E_r \,, \tag{S.380}$$

which is minimized if and only if the following equivalent cost function is minimized:

$$J' = \left(\frac{y}{E_h} - a \right)^2 \ . \tag{S.381}$$

The solution is thus to compute the correlation $y = \sum_k r_k h_k$, which can be done efficiently using a discrete-time matched filter, normalize by the energy in the impulse response, and then quantize the result to the nearest symbol in the alphabet.

(a) Here, the energy in the impulse response $h_k = \delta_k + \frac{1}{2}\delta_{k-1}$ is $E_h = 1.25$. Therefore, the ML detector with $\mathcal{A} = \{0, 1\}$ is as follows:



(b) The distance between points in the alphabet $\mathcal{A} = \{0, 1\}$ is $d = 1$. Also, after the normalization, the noise variance is σ^2/E_h. Therefore, the probability of error is:

$$P_e = Q\left(\frac{d}{2\sqrt{\sigma^2/E_h}}\right) = Q\left(\sqrt{\frac{5}{16\sigma^2}}\right) . \tag{S.382}$$

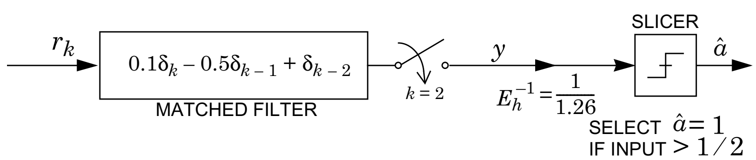
If the two symbols in $\mathcal{A} = \{0, 1\}$ are equally likely, the ML receiver is the MAP receiver, and this is the minimum probability of error.

Problem 7-10.

(a) The only difference from Problem 7-9 is that the matched filter has causal impulse response

$$h_{2-k} = 0.1\delta_k - 0.5\delta_{k-1} + \delta_{k-2} \tag{S.383}$$

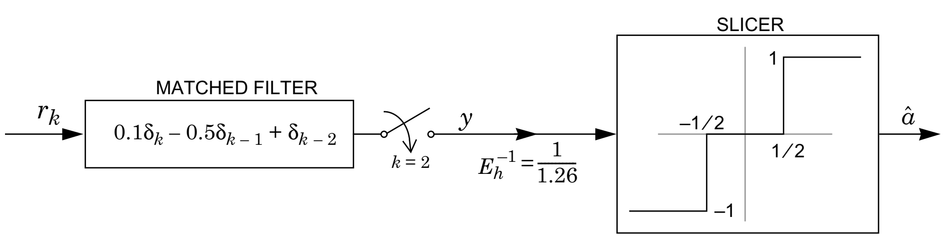
and the output is sampled at $k = 2$. The energy is thus $E_h = 1.26$. The block diagram looks like this:



(b) The probability of error is

$$Q\left(\frac{\sqrt{1.26}}{2\sigma}\right). \tag{S.384}$$

(c) For the 3-ary alphabet, the resulting receiver is shown below:



Problem 7-11.

(a) We have

$$\tilde{s}_1(t) = 0\,, \qquad \tilde{s}_2(t) = h(t) \tag{S.385}$$

$$\| \tilde{\mathbf{S}}_1 \|^2 = 0 \qquad \| \tilde{\mathbf{S}}_2 \|^2 = \rho_h(0) \quad . \tag{S.386}$$

Defining

$$K_1 = 0\,, \qquad K_2 = \left| \int_{-\infty}^{\infty} Y(t) e^{-j2\pi f_c t} h^*(t)\,dt \right| \,, \tag{S.387}$$

the ML detector chooses $\tilde{\mathbf{S}}_1$ if

$$I_0\left(\frac{K_1}{\sigma^2}\right) = I_0(0) = 1 \quad > \quad e^{-\rho_h(0)/(2\sigma^2)} I_0\left(\frac{K_2}{\sigma^2}\right) \quad . \tag{S.388}$$

Taking the logarithm of both sides will not change the inequality, so we get equivalently

$$\log I_0\left(\frac{K_2}{\sigma^2}\right) < \frac{\rho_h(0)}{2\sigma^2} \;. \tag{S.389}$$

In view of the monotonicity of $\log I_0(\,\cdot\,)$, the ML detector compares K_2 to some threshold v, where that threshold depends on the SNR. The receiver is the same as a passband PAM coherent receiver, consisting of a demodulator, matched filter, and sampler. The difference is that instead of comparing the complex-valued slicer input to the transmitted data symbols, we only evaluate the magnitude of this input (distance from the origin) and compare in effect to the magnitude of the data symbols.

(b) The data symbols must all have distinct radii in the signal constellation. Thus, for example, PSK would not work, but the ASK of this problem will work.

(c) When signal one is transmitted,

$$K_2 = |\langle \mathbf{Z}, \tilde{\mathbf{S}}_2 \rangle|\,, \tag{S.390}$$

where of course $\tilde{\mathbf{S}}_2 = \mathbf{H}$, the PAM pulse shape. The error probability in this case is

$$\Pr[\text{error} \mid \tilde{\mathbf{S}}_1 \text{ transmitted}] = Pr[K_2 > v] \;. \tag{S.391}$$

Similarly when signal two is transmitted,

$$K_2 = |e^{j\theta} \| \tilde{\mathbf{S}}_2 \|^2 + \langle \mathbf{Z}, \tilde{\mathbf{S}}_2 \rangle| \tag{S.392}$$

and the probability of error is

$$\Pr[\text{error} \mid \tilde{\mathbf{S}}_2 \text{ transmitted}] = Pr[K_2 < v] \;. \tag{S.393}$$

The get the overall probability of error we sum these two probabilities weighted by the prior probabilities of the two signals.

Problem 7-12.

(a) With respect to the signal, an isolated pulse before sampling has Fourier transform $H(f)F^*(f)$, and hence after sampling the discrete-time Fourier transform is

$$\frac{1}{T}\ \Sigma_m H(f + m/T)F^*(f + m/T)\ . \tag{S.394}$$

Note that the impulse response of the equivalent discrete-time channel is equivalent to the sampled isolated pulse response. Similarly, for the noise, after downconversion the noise has power spectrum $2S_N(f + f_c)$, and at the output of the matched filter, before sampling, $2S_N(f + f_c))\,|F(f)|^2$. After sampling, the spectrum is

$$\frac{2}{T}\Sigma_m S_N(f + f_c + m/T)\,|F(f + m/T)|^2\ . \tag{S.395}$$

(b) The discrete-time isolated pulse has Fourier transform

$$\frac{1}{T}\Sigma_m \frac{\left|H\left(f + \frac{m}{T}\right)\right|^2}{2S_N\left(f + f_c + \frac{m}{T}\right)}\ , \tag{S.396}$$

and the discrete-time noise has a power spectrum which is the same formula multiplied by two.

(c) When $S_N(f) = N_0/2$, the isolated pulse response is $S_h(e^{j\theta})/N_0$ and the noise power spectrum is $2S_h(e^{j\theta})/N_0$. If we scale the signal size by N_0, the noise spectrum is scaled by $N_0{}^2$, and we get the same answer as in the text.

Problem 7-13. Let us begin by assuming that the received pulse is of the form

$$h(t) = \sum_{m=-\infty}^{\infty} h_m g(t - mT) \tag{S.397}$$

for some pulse $g(t)$ for which $g(t)$ and $g(t - mT)$ are mutually orthogonal for $m \neq 0$. In other words, assume that $g(t)$ convolved with its *match* is a Nyquist pulse. (For example, the zero-excess-bandwidth sinc function works.) Then, the sampled autocorrelation becomes:

$$\rho_h(z) = \int_{-\infty}^{\infty} \sum_{m=-\infty}^{\infty} h_m g(t - mT) \sum_{l=-\infty}^{\infty} h_l{}^* g^*(t - lT - kT) \tag{S.398}$$

$$= E_g \sum_{m=-\infty}^{\infty} h_m h^*_{m-k}\ . \tag{S.399}$$

where $E_g = \int_{-\infty}^{\infty} |g(t)|^2 dt$. Taking the Z-transform, we get

$$S_h(z) = E_g H(z) H^*(1/z^*)\ . \tag{S.400}$$

Now this looks a lot like the factorization of the folded spectrum,

$$S_h(z) = \gamma^2 M(z) M^*(1/z^*)\ , \tag{S.401}$$

where γ^2 is the geometric mean of $S_h(z)$, and where $M(z)$ is monic and minimum phase. Comparing (S.400) and (S.401), we conclude that it suffices to choose:

$$H(z) = M(z) \tag{S.402}$$

and furthermore scaling the fundamental pulse so that its energy is equal to the geometric mean of the folded spectrum:

$$E_g = \gamma^2\ . \tag{S.403}$$

Problem 7-14. The conditional probability density is

$$p(x_1 \ldots x_N \mid s_1 \ldots s_N) = \prod_{k=1}^{N} e^{-\alpha s_k} \frac{(\alpha s_k)^{x_k}}{x_k!} \tag{S.404}$$

and hence the log-likelihood function is

$$-\log p(x_1, \ldots x_N \mid s_1, \ldots s_N) = \sum_{k=1}^{N} \left(\alpha s_k - x_k \log_e(\alpha s_k) + \log_e(x_k!) \right)\ . \tag{S.405}$$

The last term is independent of the signal, as is the $x_k \log_e \alpha$ term, so the simplified branch metric is:

$$\text{branch metric} = \left(\alpha s_k - x_k \log_e s_k \right). \tag{S.406}$$

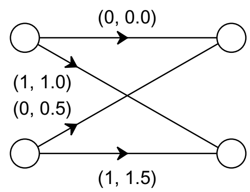
Problem 7-15. The log-likelihood function is given by the following table, where the last column specifies the branch metric:

x	y	$\Pr[y \mid x]$	$-\log \Pr[y \mid x]$
0	0	$1-p$	$-\log(1-p)$
0	1	p	$-\log(p)$
1	2	0	$-\infty$
1	0	0	$-\infty$
1	1	p	$-\log(p)$
1	2	$1-p$	$-\log(1-p)$

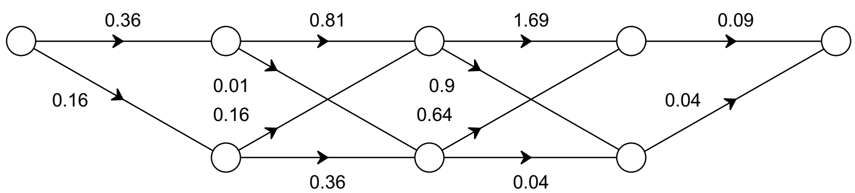
Problem 7-16.

(a) The ML detector can operate independently on each observation. It will perform a threshold test with the threshold set at 0.5. The decision is $\{\hat{x}\}_k = \{1, 1, 1, 0\}$.

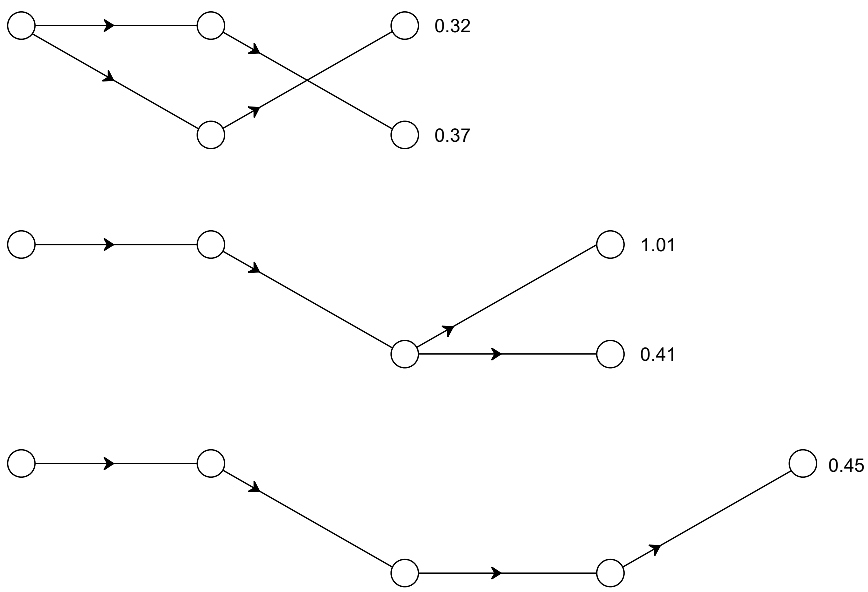
(b) The outputs associated with each transition are shown in Fig. 5-30, and repeated here:



We can find the transition weights by just subtracting those outputs from the observations and squaring. The result is shown in the following figure.



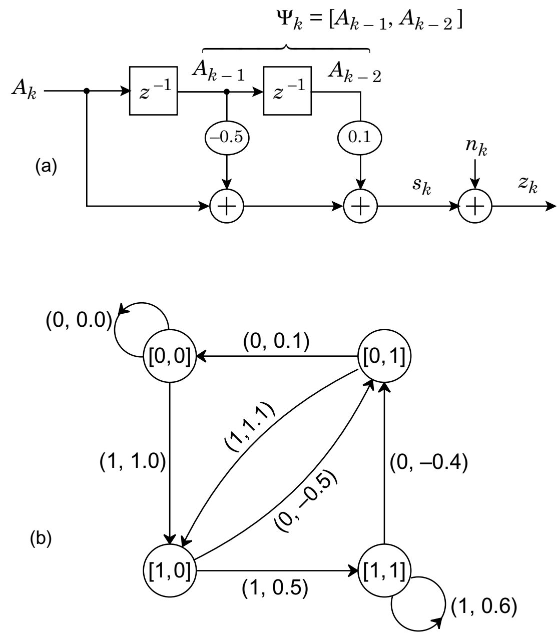
Performing the Viterbi algorithm, the surviving paths and their path metrics after each observation are shown in the following figure.



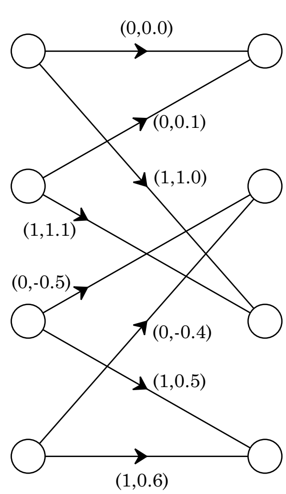
The decision is $\{\hat{x}\}_k = \{0, 1, 1, 0\}$, which is different from the decision in part (a). The knowledge of the ISI is useful.

Problem 7-17.

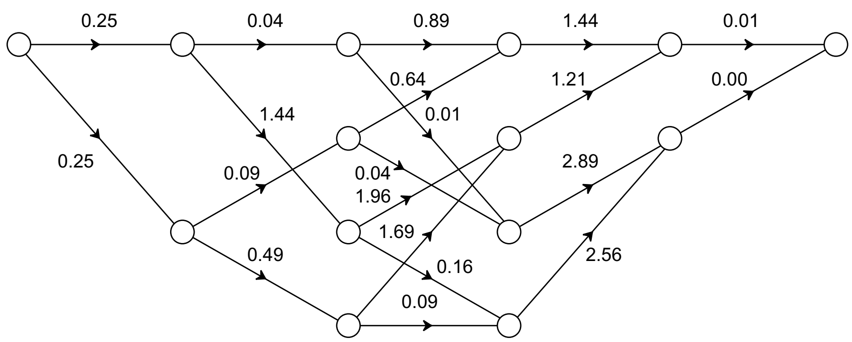
(a) The shift register model and state transition diagram are shown in the following figure:



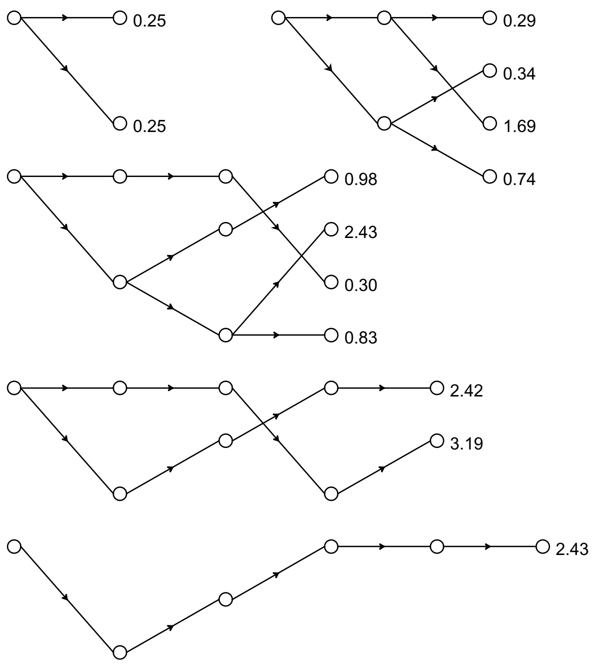
(b) One stage of the trellis is shown below:



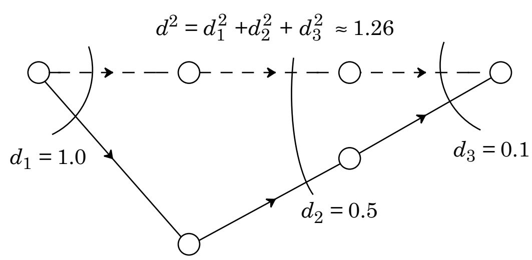
(c) Here is the complete trellis:



(d) Here is the Viterbi algorithm applied to the trellis:



(e)

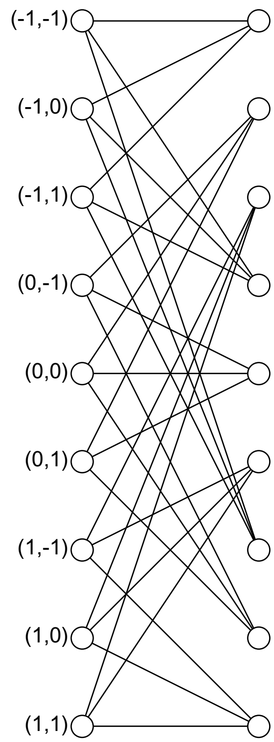


Problem 7-18.

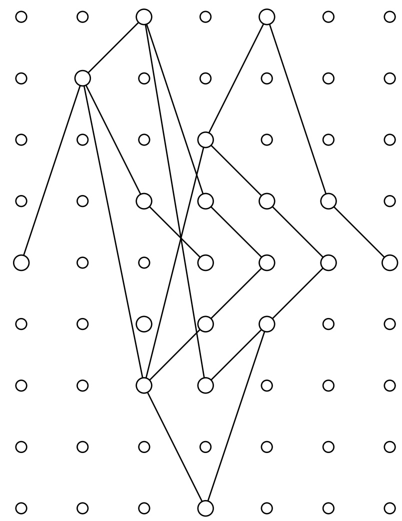
(a) The state is $(\varepsilon_{k-1}, \varepsilon_{k-2})$, and the branch metric is

$$|\varepsilon_k + m_1\varepsilon_{k-1} + m_2\varepsilon_{k-2}|^2 \quad . \tag{S.407}$$

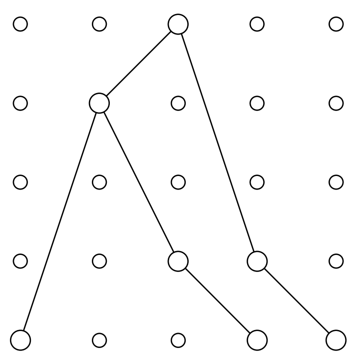
One stage of the trellis, not labeled with the branch metrics, is shown below:



(b) A finite set of error events which do not pass through the same state or its negative twice is shown below:



(c) Shown below are the single-error error event, and another error event corresponding to two symbol errors:



The shorter path has path metric $(1 + m_1{}^2 + m_2{}^2)$, while the longer path has metric

$$1 + (1 + m_1)^2 + (m_1 + m_2)^2 + m_2{}^2 \,, \tag{S.408}$$

and hence will have a smaller metric if $m_1 \approx 1$ and $m_2 \approx -m_1$. For example, if $m_1 = 1$ and $m_2 = -1$, the minimum distance is $\sqrt{2}$.

Remark: But $m_1 = 1$ and $m_2 = -1$ implies that $M(z) = 1 + z^{-1} - z^{-2}$ is not minimum phase. Does the answer change if we constrain $M(z)$ to be minimum phase?

Problem 7-19. The metric in this case is $(\varepsilon_k + d\varepsilon_{k-1} + d\varepsilon_{k-2})^2$, and the trellis diagram is pictured in Fig. S-6.

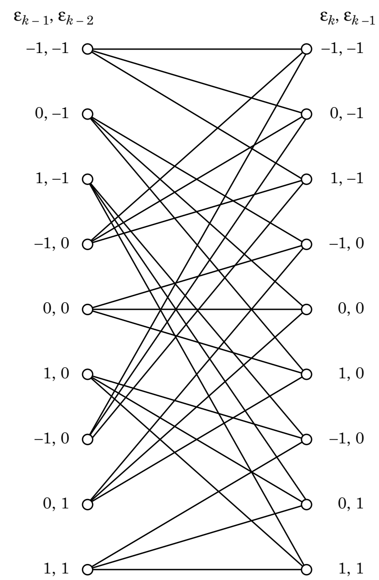


Fig. S-6. Trellis diagram for a four-state ISI channel.

A set of five paths guaranteed to include the minimum distance path is pictured in Fig. S-7.

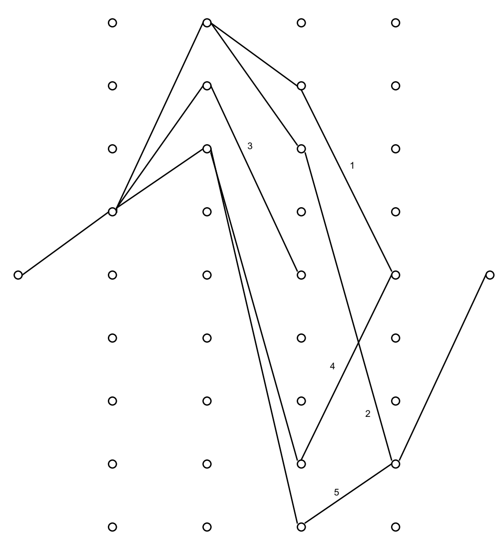


Fig. S-7. A set of candidate minimum-distance paths for the trellis of Fig. S-6.

The path metrics for these paths are:

$$\begin{array}{ll} \text{Path 1:} & 1 + (1 + d)^2 + 4d^2 + d^2 \\ \text{Path 2:} & 1 + (1 + d)^2 + (1 - 2d)^2 + d^2 \\ \text{Path 3:} & 1 + d^2 + d^2 \\ \text{Path 4:} & 1 + (1 - d)^2 + d^2 \\ \text{Path 5:} & 1 + (1 - d)^2 + 1 + 4d^2 + d^2 \end{array} \tag{S.409}$$

The metric for paths 1 and 2 are always bigger than path 3, and similarly the metric for path 5 is always bigger than path 4. When $0 \le d \le 1/2$, the path 3 metric is smaller than path 4. Thus, the answer is

$$d_{\min}{}^2 = \begin{cases} 1 + 2d^2, & 0 \le d \le 1/2 \\ 1 + d^2 + (1 - d)^2, & 1/2 \le d \le 1 \end{cases}. \tag{S.410}$$

Problem 7-20. The trellis diagram and the two error events at a minimum distance $\sqrt{2}$ are shown in Fig. S-8.

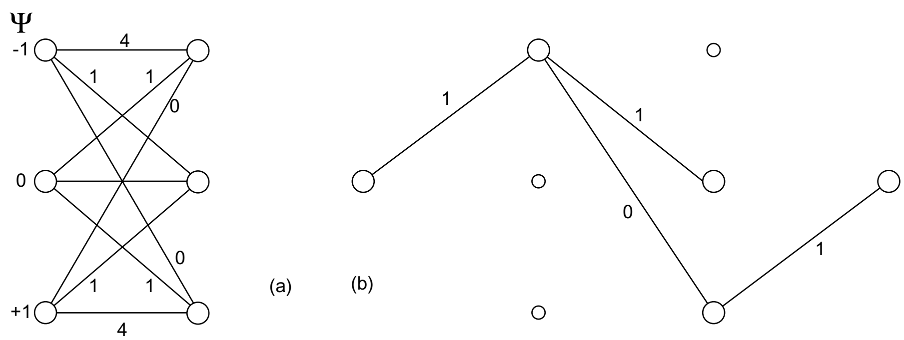
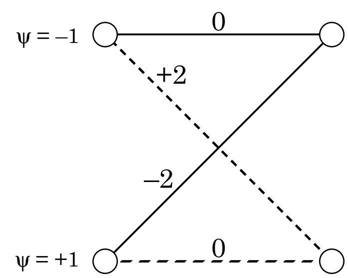


Fig. S-8. A trellis diagram and two corresponding events at the minimum distance of $\sqrt{2}$.

As in Example 7-31, there is an infinite set of error events at this minimum distance, where the only difference is that each error event corresponds to a sequence of alternating data symbols. The error probability estimates are the same. The intuitive explanation is that the output of the channel is zero during sequences of alternating data symbols. It is therefore difficult to distinguish the two sequences of alternating symbols which are the complement of each other.

Problem 7-21. Since the channel transfer function has memory $\mu = 1$, the state at time k is the previous input symbol, $\psi_k = a_{k-1}$. Since the alphabet is binary, there are only two states, $\psi_k \in \{\pm 1\}$. One stage of the trellis is shown to the right. The transitions corresponding to inputs of +1 are shown as dotted lines. (These are the transitions that terminate at the state $\psi = +1$.) The branches are labeled by the unique signal output $s^{(p,\,q)}$ associated with the corresponding state transition from state p to state q; since $H(z) = 1 - z^{-1}$, this output is simply $q - p$.



The branch metric for the transition from state p to q at time k is proportional to (7.65):

$$\gamma_k(p,\,q) = 2e^{-|y_k - s^{(p,\,q)}|^2/(2\sigma^2)}\, p_{A_k}(a^{(p,\,q)})\ , \tag{S.411}$$

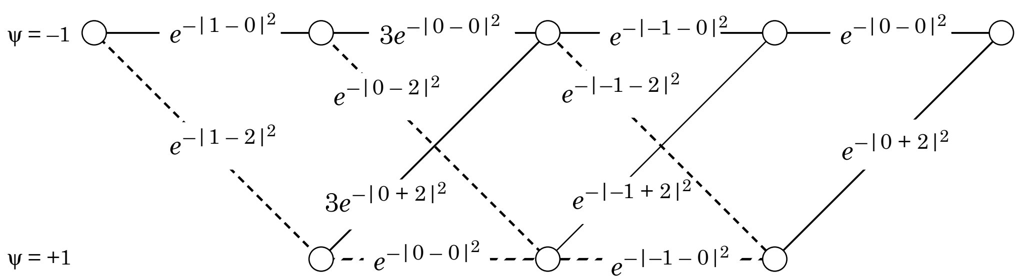
where we have dropped a factor of $1/(4\pi\sigma^2)$ to simplify calculations (see section 7.5.2). At time $k = 0$ and $k = 2$, when the symbols –1 and +1 are equally likely, this simplifies to:

$$\gamma_k(p,\,q) = e^{-|y_k - s^{(p,\,q)}|^2}\ , \tag{S.412}$$

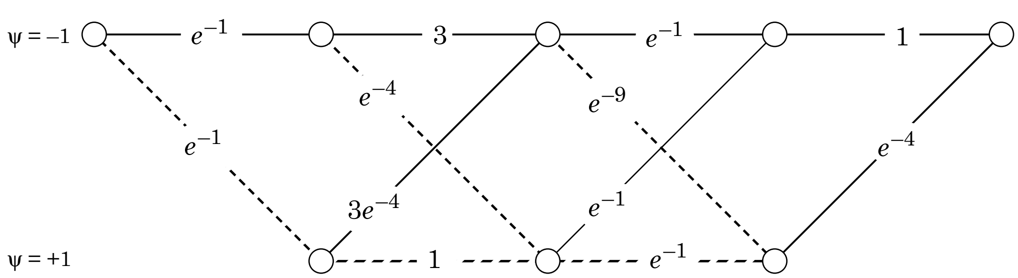
whereas at time $k = 1$ (where –1 is three times as likely as +1) it simplifies to:

$$\begin{aligned} \gamma_1(p,\,q) &= 3e^{-|y_k - s^{(p,\,q)}|^2} \quad \text{when } q = -1, \text{ and} \\ \gamma_1(p,\,q) &= \ \ e^{-|y_k - s^{(p,\,q)}|^2}, \quad \text{when } q = +1\ . \end{aligned} \tag{S.413}$$

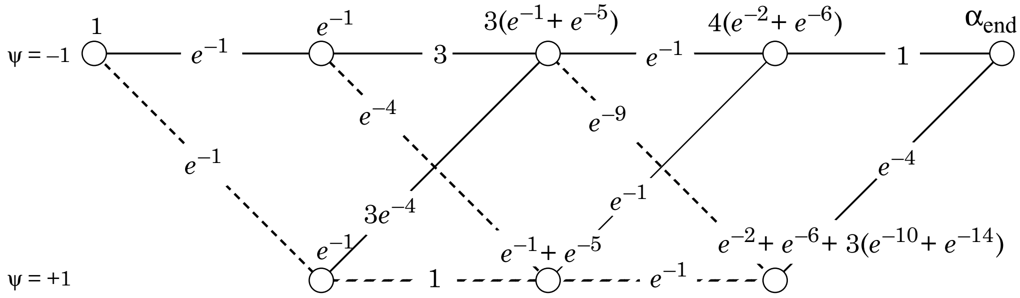
(We dropped another factor of $1/2$ here, as allowed by section 7.5.2, to simplify as much as possible.) Therefore, the entire trellis diagram with labeled branch metrics looks like this:



which simplifies to this:



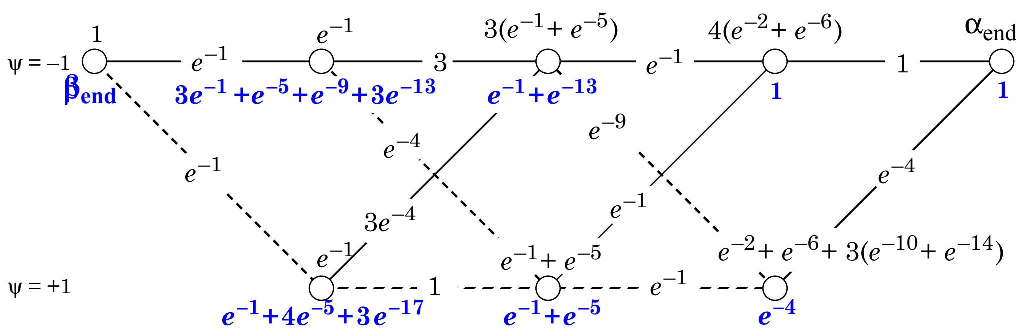
This completes step 1 of the BCJR algorithm. We can now proceed exactly as in Example 7-25. The next step is step 2, the forward recursion that calculates the forward metrics $\{\alpha_k(p)\}$. We write these metrics above the corresponding nodes of the trellis:



where to avoid clutter, we did not write the last forward metric on the trellis:

$$\begin{aligned}\alpha_{end} &= \alpha_4(-1)\\ &= 4e^{-2}+5e^{-6}+e^{-10}+3(e^{-14}+e^{-18}).\end{aligned} \tag{S.414}$$

Next comes step 3, the backward recursion that calculates the backward metrics $\{\beta_k(p)\}$. In the following we write these metrics <u>below</u> the corresponding node of the trellis, in **boldface and blue**:



where again the last metric is not shown, but can be calculated as:

$$\begin{aligned}\beta_{end} &= \beta_0(-1)\\ &= 4e^{-2}+5e^{-6}+e^{-10}+3(e^{-14}+e^{-18})\\ &= \alpha_{end}\,.\end{aligned} \tag{S.415}$$

(This is a good sanity check: If the last backward metric does not match the last forward metric, you must have made an arithmetic mistake somewhere.)

Therefore, from (7.77):

$$\lambda_0 = \log\left(\frac{1\times e^{-1}\times(\boldsymbol{e^{-1}+4e^{-5}+3e^{-17}})}{1\times e^{-1}\times(\boldsymbol{3e^{-1}+e^{-5}+e^{-9}+3e^{-13}})}\right)$$

$$= \log\left(\frac{1+4e^{-4}+3e^{-16}}{3+e^{-4}+e^{-8}+3e^{-12}}\right) = -1.034\ ,$$

$$\lambda_1 = \log\left(\frac{e^{-1}\times e^{-4}\times(\boldsymbol{e^{-1}+e^{-5}})+e^{-1}\times 1\times(\boldsymbol{e^{-1}+e^{-5}})}{e^{-1}\times 3\times(\boldsymbol{e^{-1}+e^{-13}})+e^{-1}\times 3e^{-4}\times(\boldsymbol{e^{-1}+e^{-13}})}\right)$$

$$= \log\left(\frac{1}{3}\left(\frac{1+e^{-4}}{1+e^{-12}}\right)\right) = -1.080\ , \tag{S.416}$$

$$\lambda_2 = \log\left(\frac{3(e^{-1}+e^{-5})\times e^{-9}\times\boldsymbol{e^{-4}}+(e^{-1}+e^{-5})\times e^{-1}\times\boldsymbol{e^{-4}}}{3(e^{-1}+e^{-5})\times e^{-1}\times(\boldsymbol{1})+(e^{-1}+e^{-5})\times e^{-1}\times(\boldsymbol{1})}\right)$$

$$= \log\left(\frac{e^{-4}+3e^{-12}}{4}\right) = -5.385\ . \tag{S.417}$$

Note that a reasonable approximation can be found by keeping only the dominant terms:

$$\lambda_0 \approx \lambda_1 \approx \log(1/3) = -1.099, \qquad \lambda_2 \approx \log\left(\frac{e^{-4}}{4}\right) = -5.386\ . \tag{S.418}$$

The fact that all three λ_k values are negative indicates that all three MAP decisions will be -1.

Problem 7-22.

(a) The logarithm of the moment generation function is, from (3.158),

$$\Psi_i(v) = \int_{-\infty}^{\infty}(\lambda_i(t)+\lambda_{\text{dark}})\left[e^{vf(-t)}-1\right]dt \tag{S.419}$$

and hence the Chernov bound is

$$P_e \le \exp\left\{\frac{1}{2}\left(\int_{-\infty}^{\infty}(\lambda_1(t)+\lambda_{\text{dark}})[e^{vf(-t)}-1]+(\lambda_2(t)+\lambda_{\text{dark}})[e^{-vf(-t)}-1]\right)dt\right\}. \tag{S.420}$$

(b) By the method of variations, we substitute $f+\varepsilon\Delta f$ for f, differentiate w.r.t. ε, set $\varepsilon = 0$, and set the result to zero. The result is

$$f(-t) = \frac{1}{2}\left[\log(\lambda_2(t)+\lambda_{\text{dark}}) - \log(\lambda_1(t)+\lambda_{\text{dark}})\right]\ , \tag{S.421}$$

which says that we correlate the shot noise against the logarithm of the known intensity.

(c) Substituting into the Chernov bound from(b)., we get

$$P_e \le \exp\left\{\int_{-\infty}^{\infty}(\lambda_1(t)+\lambda_{\text{dark}})^{1/2}(\lambda_2(t)+\lambda_{\text{dark}})^{1/2}dt \;-\; \frac{E_1+E_2}{2}\right\} \tag{S.422}$$

where E_i is the energy corresponding to the intensity,

$$E_i = \int_{-\infty}^{\infty} (\lambda_1(t) + \lambda_{\text{dark}})\, dt\ . \tag{S.423}$$

Problem 7-23. (Typo: The left-hand side of (7.169) should be V_m, not V_l.) First equate the two representations of $f_l(t)$, (7.128) and (7.46),

$$f_l(t) = \sum_{k=1}^{N} F_k^{(l)} \psi_k(t) = \sum_{i=1}^{\infty} \frac{s_{l,i}}{\sigma_i} \phi_i(t)\ \ . \tag{S.424}$$

Now form the inner product of both sides with $\phi_m(t)$,

$$\sum_{k=1}^{N} F_k^{(l)} \int_0^T \psi_k(t)\phi_m{}^*(t)\ dt = \sum_{i=1}^{\infty} \frac{s_{l,i}}{\sigma_i} \int_0^T \phi_i(t)\phi_m{}^*(t)\, dt \tag{S.425}$$

and applying (7.130) we get

$$\sum_k F_k^{(l)} \psi_{k,m} = \frac{s_{l,m}}{\sigma_m}\ \ . \tag{S.426}$$

Finally, substituting into (7.44) yields:

$$\begin{aligned} V_l &= \sum_{i=1}^{\infty} Y_i \frac{s_{l,i}{}^*}{\sigma_i} \\ &= \sum_{i=1}^{\infty} \frac{Y_i}{\sigma_i} \sum_{k=1}^{N} F_k^{(l)} \psi_{k,i}{}^* \\ &= \sum_{k=1}^{N} F_k^{(l)*} \sum_{i=1}^{\infty} \frac{Y_i}{\sigma_i} \cdot \psi_{k,i}{}^* \\ &= \sum_{k=1}^{N} F_k^{(l)*} U_k\ . \end{aligned} \tag{S.427}$$

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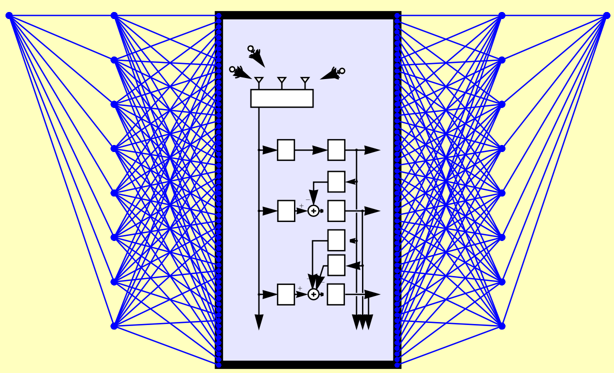

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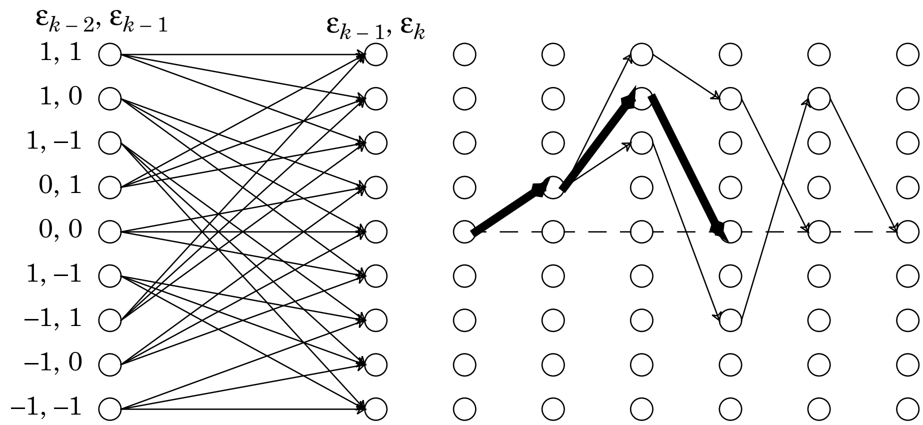
Chapter 8

Problem 8-1. The trellis and three error events are sketched below:



The error event with the dark lines is the one requested in (b), and the two events in light lines are those requested in (c). (The answer to (c) is not unique.)

Problem 8-2. Let μ be the order of the FIR filter $M(z)$. We will show that the inequality (8.31) is strict for every error event. To do this, we need to show that at least one of the terms thrown away ($m \geq 2$) is non-zero, for any error event. Assume is the length of a given error event ($\varepsilon_L \neq 0$ and $\varepsilon_k = 0$ for $k > L$), and consider the $m = L + \mu$ term,

$$\left| \sum_{k=1}^{L} \varepsilon_k m_{L+\mu-k} \right|^2 = |\varepsilon_L m_\mu|^2 > 0 \quad . \tag{S.428}$$

Thus, the inequality is strict for each and every error event, which implies that it is strict for the minimum-distance error event.

Problem 8-3. Since

$$|H|^{-2} = (1 - cz^{-1})(1 - c^* z) \quad , \tag{S.429}$$

the coefficient of z^0 is $1 + |c|^2$, independent of whether the channel is minimum-phase or maximum-phase. For $|c| < 1$, the geometric mean is clearly unity, since (S.429) is in the form of a minimum-phase spectral factorization. When $|c| > 1$, we can rewrite (S.429) in the form

$$|H|^{-2} = |c|^2 (1 - (c^*)^{-1} z^{-1})(1 - c^{-1} z) \quad , \tag{S.430}$$

and thus the geometric mean is $|c|^2$.

Problem 8-4.

(a) $(1 - cz^{-1})$

$$\varepsilon^2_{\text{ZF-LE}} = N_0 \langle\, |H|^{-2} \rangle_A = N_0 (1 + \,|c\,|^2) \tag{S.431}$$

from Problem 8-3.

(b) When $c = 0$ there is no noise enhancement. As $|c\,| \to 1$ the noise enhancement approaches 3 dB. For $|c\,| > 1$ the noise enhancement gets even larger because the channel transfer function gets smaller.

(c) The maximum-phase case will not arise in practice, because the channel impulse response would be both IIR and anticausal.

Problem 8-5. For the minimum-phase case, multiply the transfer function by $\sqrt{1 - |c|^2}$ to normalize it, and thus

$$\varepsilon^2_{\text{ZF-LE}} = N_0 \frac{1 + |c|^2}{1 - |c|^2} \ . \tag{S.432}$$

For the maximum-phase case, the normalization constant can be determined by

$$H(z) = \frac{z}{c(1 - c^{-1}z)} = \frac{1}{c} \sum_{k=0}^{\infty} c^{-k} z^{k+1} \ , \tag{S.433}$$

and the energy is $1/(|c\,|^2 - 1)$ so that the normalization constant becomes $\sqrt{|c|^2 - 1}$, and

$$\varepsilon^2_{\text{ZF-LE}} = N_0 \frac{|c|^2 + 1}{|c|^2 - 1} \tag{S.434}$$

The solution is quite different from Problem 8-4, since the noise enhancement approaches infinity as $|c\,| \to 1$ and goes away as $|c\,|$ gets large. This is to be expected, since as $|c\,| \to 1$ the channel transfer function on the unit circle goes to zero at all frequencies except the pole location, and as $|c\,|$ gets large the channel approaches a negative unit delay, which is easily equalized without noise enhancement by a unit delay.

Problem 8-6.

(a) From Problem 8-3, the geometric mean is unity and hence $\varepsilon^2_{\text{ZF-DFE}} = N_0$. No precursor equalizer is required, and the postcursor equalizer is

$$H(z) - 1 = \frac{cz^{-1}}{1 - cz^{-1}} \ . \tag{S.435}$$

(b) From Problem 8-3, the geometric mean is $|c\,|^2$, and hence $\varepsilon^2_{\text{ZF-DFE}} = N_0 |c\,|^2$. Writing H in monic form,

$$H(z) = \frac{-c^{-1}z}{1 - c^{-1}z} \tag{S.436}$$

we get $r = 1$, $H_0 = -c^{-1}$, and $H_{\max} = 1/(1 - c^{-1}z)$. The precursor equalizer is

$$C(z)E(z) = \frac{-c(1 - (c^*)^{-1}z^{-1})}{1 - c^{-1}z} \tag{S.437}$$

and the postcursor equalizer is $E(z) - 1 = -(c^*)^{-1}z^{-1}$.

(c) In the maximum-phase case the precursor equalizer is IIR and anticausal, and hence not practical to implement. However, the MSE gets smaller because the equalizer utilizes the large delayed sample for decision making. The MSE of the LE-ZF is always larger than the LE-ZF, by a factor of $1 + |c|^2$ for $|c| < 1$ and by a factor of $(1+|c|^2)/|c|^2$ for $|c| > 1$. This difference is largest (about 3 dB) in the region of $|c| \approx 1$.

Problem 8-7. In the minimum-phase case the normalization constant is $\sqrt{1-|c|^2}$, and

$$\varepsilon^2_{\text{ZF- DFE}} = \frac{N_0}{1-|c|^2} \, . \tag{S.438}$$

In the maximum-phase case the normalization constant is $\sqrt{|c|^2-1}$ and

$$\varepsilon^2_{\text{ZF- DFE}} = N_0 \frac{|c|^2}{|c|^2-1} \, . \tag{S.439}$$

As for the LE-ZF, the MSE blows up as the pole approaches the unit circle. The DFE-ZF can tolerate zeros on the unit circle, but not poles. Likewise, the noise enhancement goes away as $|c| \to \infty$ because the equalizer bases its decision on the larger delayed sample, which is asymptotically unity in the normalized case.

Problem 8-8. The formula for the MSE follow readily. The spectral factorization of (8.84) approaches, as $S_Z \to 0$,

$$S_Y \to S_A HH^* \tag{S.440}$$

$$S_Y \to |H_0|^2 A_a{}^2 G_a G_a{}^* H_{\min} H_{\max}{}^* H_{\min}{}^* H_{\max} \tag{S.441}$$

$$A_y{}^2 \to |H_0|^2 \sigma_A{}^2 \tag{S.442}$$

$$G_y \to G_a H_{\min} H_{\max}{}^* \tag{S.443}$$

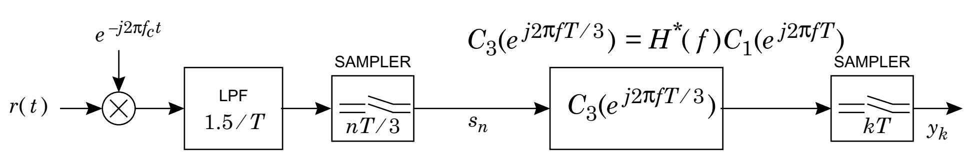
and finally

$$CE \to \frac{H^*_{\max}}{H_0 H_{\max}} G_z{}^{-1} \tag{S.444}$$

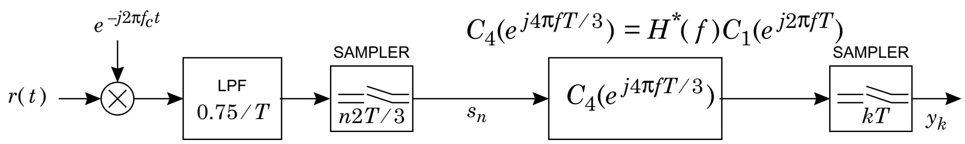
which is the DFE-ZF precursor equalizer.

Problem 8-9.

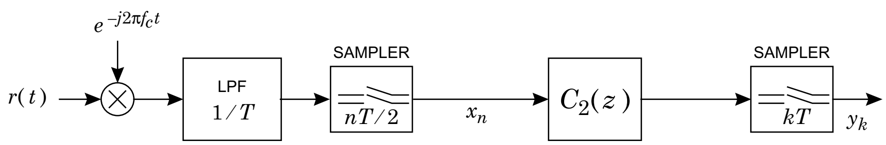
(a) When the excess bandwidth is less than 200%, we get the following picture:



(b) When the excess bandwidth is less than 50%, we get the following picture



Problem 8-10. The picture is as follows:



The output is

$$y_k = \sum_{m=-\infty}^{\infty} c_m x_{2k-m} \ . \tag{S.445}$$

Problem 8-11. Given a baseband transmit spectrum S_X, the transmit power constraint of (8.117) is

$$P/2 = |F| \cdot \langle S_X \rangle_{\{A,\, F\}} \ , \tag{S.446}$$

where $S_x = L - S_N / |H|^2$ for $f \in F$, and hence we get a relation for L,

$$P/2 = |F| \cdot (L - \langle S_N / |H|^2 \rangle_{\{A,\, F\}}) \ . \tag{S.447}$$

Substituting this into (8.115), if the integral is restricted to $f \in F$,

$$\begin{aligned} 2^{C/|F|} &= \langle 1 + S_X |H|^2 / S_N \rangle_{\{G,\, F\}} \\ &= \frac{L}{\langle S_N / |H|^2 \rangle_{G,\, F}} \\ &= \frac{\frac{P}{2}|F| + \langle S_N / |H|^2 \rangle_{A,\, F}}{\langle S_N / |H|^2 \rangle_{G,\, F}} \ . \end{aligned} \tag{S.448}$$

Finally,

$$C = |F| \cdot \log_2\left(\frac{\frac{P}{2}|F| + \langle S_N / |H|^2 \rangle_{A,\, F}}{\langle S_N / |H|^2 \rangle_{G,\, F}} \right) \ . \tag{S.449}$$

Problem 8-12. The pulse at the output of the receive filter has Fourier transform:

$$TG(f)H(f)F(f) \tag{S.450}$$

and the noise has power spectrum $TS_N(f)|F(f)|^2$. After sampling, the isolated pulse has Fourier transform

$$H(e^{j2\pi fT}) = \sum_m G\left(f - \frac{m}{T}\right) H\left(f - \frac{m}{T}\right) F\left(f - \frac{m}{T}\right) \quad , \tag{S.451}$$

and the noise has power spectrum

$$S_N(e^{j2\pi fT}) = \sum_m S_N\left(f - \frac{m}{T}\right) \left| F\left(f - \frac{m}{T}\right) \right|^2 \quad , \tag{S.452}$$

(a) The capacity is given by (8.122) with these values of $H(e^{j2\pi fT})$ and $S_N(e^{j2\pi fT})$.

(b) For this case, $F = G^*H^*$, and thus

$$H(e^{j2\pi fT}) = \sum_m \left| G\left(f - \frac{m}{T}\right) \right|^2 \cdot \left| H\left(f - \frac{m}{T}\right) \right|^2 , \tag{S.453}$$

and

$$S_N(e^{j2\pi fT}) = \sum_m S_N\left(f - \frac{m}{T}\right) \left| G\left(f - \frac{m}{T}\right) \right|^2 \cdot \left| H\left(f - \frac{m}{T}\right) \right|^2 \quad . \tag{S.454}$$

Note that when the noise is white, $H(e^{j2\pi fT})$ and $S_N(e^{j2\pi fT})$ have the same shape; that is, they are equal within a multiplicative constant N_0. Thus,

$$\frac{S_N(e^{j2\pi fT})}{|H(e^{j2\pi fT})|^2} = \frac{N_0}{|H(e^{j2\pi fT})|} \, . \tag{S.455}$$

(c) First, the discrete-time system has to be able to generate the water-pouring spectrum. A sufficient condition for this is the following: If F is the water-pouring band, which must be symmetric about $f = 0$, then the sampling rate is twice $|F|/2$, or $1/T > |F|$, and the transmit filter is ideally bandlimited to half the sampling rate and non-zero over this bandwidth. (This is not a necessary condition, because if F is a "generalized Nyquist interval" with respect to sampling rate $1/T$, then the water-pouring spectrum can be generated by an appropriate transmit filter. A generalized Nyquist interval has the property that for each $|f| \le 1/(2T)$, $f + m/T \in F$ for precisely one value of m.) Second, the receive filter must prevent aliasing and allow all frequencies within the water-pouring band to pass. A sufficient condition for this is that it be an ideal LPF bandlimited to $1/(2T)$ Hz. (Again this is not a necessary condition. If the water-pouring band is a generalized Nyquist interval with respect to sampling rate $1/T$ then the receive filter that is ideally bandlimited to this water-pouring band will do.)

(d) For this case, if the transmit filter and sampling rate meet the criteria of (c), then the receive filter will automatically be OK. The fact that the receive filter is not flat within the generalized Nyquist interval will not be a problem, since this transfer function can always be reversed with an equalizer filter in a reversible fashion.

(e) The precursor equalizer is a reversible operation, and thus will not affect the capacity.

Problem 8-13. For the baseband case, we get $S_N = N_0/2$ and $H = 1$, and thus from (8.123), $S_X = L - N_0/2$. From the power constraint, $P = (L - N_0/2)/T$ or $S_X = PT$. Substituting into (8.122),

$$C = \frac{1}{2T}\log_2\,(1 + 2PT/N_0) = B\log_2(1 + P/N_0B)\ , \tag{S.456}$$

since in this case the sampling rate is $1/T = 2B$.

In the passband case, we have $S_Z = N_0$ and $H = 1$, and thus from(8.127) $S_X = 2L - N_0/2$, and the power constraint of (8.125) becomes $P = (2L - N_0)/T$. Substituting into (8.126),

$$C = \frac{1}{T}\log_2(1{+}PT/N_0) = B\log_2\,(1{+}\,P/N_0B)\ , \tag{S.457}$$

since the sampling rate is $1/T = B$ in this case.

Problem 8-14.

(a) At the output of the channel, the noise spectrum is S_Z and hence the total noise power is $|F|\langle S_Z\rangle_{A,\,F}$. If the channel input signal has power spectrum S_X, confined to water-pouring band F, then the total signal power at the channel output is:

$$|F|\cdot\langle S_X|H|^2\rangle_{A,\,F}. \tag{S.458}$$

This is easily related back to the transmit signal power P through the water-pouring spectrum, since for $f \in F$,

$$S_X|H|^2 = |H|^2\Big(\frac{P}{|F|} + \langle S_Z/\,|H|^2\rangle_{A,\,F}\Big) - S_Z\ , \tag{S.459}$$

and from this the channel-output SNR is

$$\begin{aligned} SNR_{\text{out}} &= \frac{|F|\langle S_X|H|^2\rangle_{A,\,F}}{|F|\langle S_Z\rangle_{A,\,F}} \\ &= \langle\,|H|^2\rangle_{A,\,F}\Big(\frac{P}{|F|} + \langle S_Z/\,|H|^2\rangle_{A,\,F}\Big)\frac{\langle|H|^2\rangle_{A,\,F}}{\langle S_Z\rangle_{A,\,F}} - 1\ . \end{aligned} \tag{S.460}$$

(b)

$$SNR_{\text{norm}} = \frac{(SNR_{\text{out}} + 1)\dfrac{\langle S_Z\rangle_{A,\,F}}{\langle|H|^2\rangle_{A,\,F}} - \Big\langle\dfrac{S_Z}{|H|^2}\Big\rangle_{A,\,F}}{2^{\nu B_0/F}\Big\langle\dfrac{S_Z}{|H|^2}\Big\rangle_{G,\,F} - \Big\langle\dfrac{S_Z}{|H|^2}\Big\rangle_{A,\,F}}\ . \tag{S.461}$$

Problem 8-15. We require that $\langle\,|G|^2\rangle_A = PT/E_a$. The effect of the transmit filter is to change the channel from H to GH, and hence the MSE at the output of the DFE-ZF to

$$\varepsilon^2_{\text{ZF- DFE}} = \langle S_Z/\,|GH|^2\rangle_G = \frac{\langle S_Z/|H|^2\rangle_G}{\langle|G|^2\rangle_G}\ . \tag{S.462}$$

Using the geometric mean inequality $\langle\,|G|^2\rangle_G \le \langle\,|G|^2\rangle_A = PT/E_a$, we get:

$$\varepsilon^2_{\text{ZF- DFE}} \ge \langle S_Z/\,|H|^2\rangle_G E_a/PT\,, \tag{S.463}$$

with equality if and only if $G = T\sqrt{P/E_a}$.

Problem 8-16. We have that $\langle S_A \rangle_A = E_a$ and constraint $\langle S_A \,|\, G \,|^2 \rangle_A = PT$.

(a) Expliciting calculating the MSE,

$$\begin{aligned}\langle S_Z / \,|\, GH \,|^2 \rangle_G &= \langle S_A S_Z / S_A \,|\, G \,|^2 \,|\, H \,|^2 \rangle_G \\ &= \frac{\langle S_Z / |H|^2 \rangle_G \langle S_A \rangle_G}{\langle S_A |G|^2 \rangle_G} \\ &\geq \langle S_Z / \,|\, H \,|^2 \rangle_G \frac{\langle S_A \rangle_G}{PT} \quad . \end{aligned} \tag{S.464}$$

Thus, the MSE is bounded below by a quantity that can be achieved when $S_A \,|\, G \,|^2$ is a constant, namely $\sqrt{PT}$.

(b) $\varepsilon^2_{\text{ZF- DFE}} = \langle S_Z / \,|\, H \,|^2 \rangle_G \langle S_A \rangle_G / (PT)$.

(c) Since $\langle S_A \rangle_G \leq \langle S_A \rangle_A = E_a$, we get that, when the transmit filter is optimized,

$$\varepsilon^2_{\text{ZF- DFE}} \leq \langle S_Z / \,|\, H \,|^2 \rangle_G \frac{E_a}{PT} \quad . \tag{S.465}$$

The right side of (S.465) is the MSE for the white symbol case, established in Problem 8-15.

Problem 8-17. The derivation for the error probability assumed Gaussian noise at the slicer. For the DFE-MSE, the slicer error includes residual ISI, and hence is not Gaussian. It would be surprising to find the SNR gap to capacity shrunk by the presence of ISI, so it is likely that if the effect of ISI at the slicer were taken into account we would find ISI to be beneficial.

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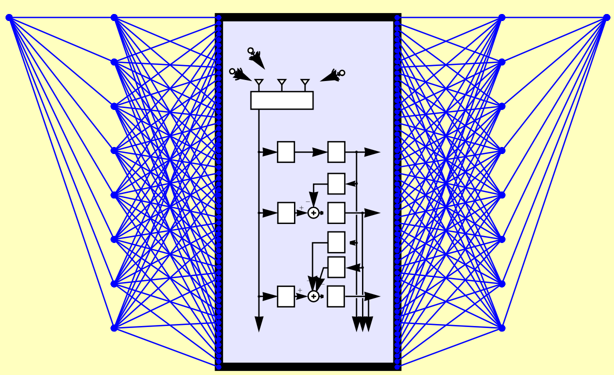

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Kluwer Academic Publishers

Chapter 9

Problem 9-1. For a predictor with coefficient vector $\mathbf{f}$, the error is given by

$$\begin{aligned} E[\,|E_k|^2] &= E[\,|R_k - \mathbf{f}'\boldsymbol{r}_k|^2] \\ &= E[\,|R_k|^2] - 2\mathrm{Re}\{\mathbf{f}^* E[R_k \bar{\boldsymbol{r}}_k]\} + \mathbf{f}^* E[\bar{\boldsymbol{r}}_k \boldsymbol{r}_k^T]\mathbf{f} \\ &= \phi_0 - 2\mathrm{Re}\{\mathbf{f}^*\alpha\} + \mathbf{f}^*\Phi\mathbf{f} \end{aligned} \tag{S.466}$$

where

$$\alpha = E[R_k \bar{\boldsymbol{r}}_k] = [\phi_1, \phi_2, \ldots \phi_n]^T, \tag{S.467}$$

$$\Phi = E[\bar{\boldsymbol{r}}_k \boldsymbol{r}_k^T], \tag{S.468}$$

the same as before, by wide-sense stationarity. The solution is the same as the the equalizer, with the new definition of α.

Problem 9-2. The orthogonality principle of (9.26) implies that

$$\mathbf{0} = E[E_k \bar{\boldsymbol{r}}_k] = E[(A_k - \boldsymbol{c}^T\boldsymbol{r}_k)\bar{\boldsymbol{r}}_k] = \alpha - \Phi\boldsymbol{c} \tag{S.469}$$

where the last equality follows since $\boldsymbol{c}^T\boldsymbol{r}_k$ is a scaler and therefore

$$E[(\boldsymbol{c}^T\boldsymbol{r}_k)\bar{\boldsymbol{r}}_k] = E[\bar{\boldsymbol{r}}_k \boldsymbol{r}_k^T]\boldsymbol{c}\,. \tag{S.470}$$

Problem 9-3.

$$\begin{aligned} E[E_k \bar{\boldsymbol{r}}_k] &= E[(R_k - \mathbf{f}_{\mathrm{opt}}{}^T\boldsymbol{r}_k)\bar{\boldsymbol{r}}_k] \\ &= E[R_k \bar{\boldsymbol{r}}_k] - E[\bar{\boldsymbol{r}}_k \boldsymbol{r}_k^T]\mathbf{f}_{\mathrm{opt}} \\ &= \alpha - \Phi\mathbf{f}_{\mathrm{opt}} = \mathbf{0}\,. \end{aligned} \tag{S.471}$$

Problem 9-4.

(a) This follows from the definition of matrix multiplication since the element of Φ in row i and column j is $\phi_{i-j} = \phi_{j-i}$.

(b) As $L \to \infty$,

$$\sum_{i=-\infty}^{\infty} \phi_{j-i} v_i = \lambda v_j \;, \quad -\infty < j < \infty \;. \tag{S.472}$$

Since this is a convolution sum, the Fourier Transform gives:

$$S(e^{j\theta})V(e^{j\theta}) = \lambda V(e^{j\theta})\,. \tag{S.473}$$

(c) Either $V(e^{j\theta}) = 0$ or $\lambda = S(e^{j\theta})$. Since $S(e^{j\theta})$ is single valued, $\lambda = S(e^{j\theta})$ can occur at only one θ_0 since λ is a constant. Hence $V(e^{j\theta})$ will be zero at other θ. $V(e^{j\theta}) = \delta(\theta - \theta_0)$ will be an eigenvector, or $v_i = e^{j\theta_0 i}$ (a complex exponential).

(d) As $L \to \infty$ the eigenvalues are by this argument the values of the function $S(e^{j\theta})$.

Problem 9-5.

(a) Taking the Fourier transform of the autocorrelation function ϕ_k, the power spectrum is

$$\begin{aligned} S(e^{j\theta}) &= \textstyle\sum_k \alpha^{|k|} e^{jk\theta} \\ &= \frac{1-\alpha^2}{(1-\alpha e^{j\theta})(1-\alpha e^{-j\theta})} \\ &= \frac{1-\alpha^2}{1+\alpha^2 - 2\alpha\cos(\theta)}. \end{aligned} \tag{S.474}$$

(b) Assume $0 < \alpha < 1$. Then the minimum of the power spectrum is at $\cos(\theta) = -1$, and

$$\lambda_{\min} \to \frac{1-\alpha}{1+\alpha} \tag{S.475}$$

Similarly the maximum is at $\cos(2\pi f) = +1$, and

$$\lambda_{\max} \to \frac{1+\alpha}{1-\alpha} \tag{S.476}$$

(c) For $N = 2$, the autocorrelation matrix is

$$\Phi = \begin{bmatrix} 1 & \alpha \\ \alpha & 1 \end{bmatrix}, \tag{S.477}$$

and setting the determinant of $(\lambda\mathbf{I} - \Phi)$ equal to zero, we get the pair of eigenvalues $\lambda_1 = 1 - \alpha$ and $\lambda_2 = 1 + \alpha$.

(d) As $N \to \infty$ we get that

$$\frac{\lambda_{\max}}{\lambda_{\min}} = \left(\frac{1+\alpha}{1-\alpha}\right)^2 \tag{S.478}$$

and for $N = 2$,

$$\frac{\lambda_{\max}}{\lambda_{\min}} = \left(\frac{1+\alpha}{1-\alpha}\right). \tag{S.479}$$

For $\alpha \approx 1$ these values are the same.

(e) As $N \to \infty$,

$$\beta_{opt} = \frac{1-\alpha^2}{1+\alpha^2}, \tag{S.480}$$

and the dominant mode of convergence is proportional to

$$\left(\frac{2\alpha}{1+\alpha^2}\right)^k. \tag{S.481}$$

As $\alpha \to 1$, the input samples become perfectly correlated and the convergence of the MSEG algorithm slows.

Problem 9-6. The error vector is given by (9.32),

$$\begin{aligned} \boldsymbol{q}_j &= \left(\mathbf{I} - \beta\Phi\right)^j \boldsymbol{q}_0 \\ &= \sum_{i=1}^{n} \left(1 - \beta\lambda_i\right)^j \boldsymbol{v}_i \boldsymbol{v}_i{}^* \boldsymbol{q}_0 \\ &= \sum_{i=1}^{n} \gamma_{i,j} \boldsymbol{v}_i \end{aligned} \tag{S.482}$$

where

$$\gamma_{i,j} = \left(1 - \beta\lambda_i\right)^j \boldsymbol{v}_i{}^* \boldsymbol{q}_0\,. \tag{S.483}$$

The component of the error in the direction of each eigenvector $\boldsymbol{v}_i$ is $\gamma_{i,j}$, and decreases exponentially as $\left(1 - \beta\lambda_i\right)^j$. The component of the initial error in the direction of $\boldsymbol{v}_i$ is $\boldsymbol{v}_i{}^*\boldsymbol{q}_0$, the component of the initial error in the direction of $\boldsymbol{v}_i$.

Problem 9-7. From Problem 9-6,

$$\begin{aligned} \boldsymbol{q}_j{}^* \boldsymbol{v}_l &= \sum_{i=0}^{n} \gamma_{i,j} \boldsymbol{v}_i{}^* \boldsymbol{v}_l \\ &= \gamma_{l,j} \end{aligned} \tag{S.484}$$

or

$$\begin{aligned} E[E_k{}^2] - E\left[E_k{}^2\right]_{\min} &= \sum_{i=1}^{n} \lambda_i \left(\boldsymbol{q}_j{}^* \boldsymbol{v}_i\right)^2 \\ &= \sum_{i=1}^{n} \lambda_i \gamma_{i,j}{}^2 \\ &= \sum_{i=1}^{n} \lambda_i \left(1 - \beta\lambda_i\right)^{2j} \left(\boldsymbol{v}_i{}^* \boldsymbol{q}_0\right)^2 \end{aligned} \tag{S.485}$$

which decreases exponentially with n modes as $\left(1 - \beta\lambda_i\right)^{2j}$.

Problem 9-8.

(a) Assume the dominant mode is i, λ_i, $\boldsymbol{v}_i$. From Problem 9-7 , the excess MSE for this mode is

$$\text{xsmse} = \lambda_i (1 - \beta\lambda_i)^{2j} (\boldsymbol{v}_i{}^* \boldsymbol{q}_0)^2\,, \tag{S.486}$$

so that

$$10\log_{10}(\text{xsmse}) = \gamma_1 - j\gamma_2 \tag{S.487}$$

where

$$\gamma_1 = 10\log_{10}(\lambda_i(\boldsymbol{v}_i{}^*\boldsymbol{q}_0)^2) \tag{S.488}$$

$$\gamma_2 = -20\log_{10}(1 - \beta\lambda_i) \tag{S.489}$$

Note that the excess MSE measured in dB decreases linearly with time. Hence the speed of convergence is often measured in dB/sec. or some equivalent units.

(b) From Fig. 9-4, when β is small the dominant mode corresponds to $\lambda_{\min}$, so that

$$\begin{aligned}\gamma_1 &= 10\log_{10}(\boldsymbol{v}_i{}^*\boldsymbol{q}_0)^2 + 10\log_{10}\lambda_{\min} \\ \gamma_2 &= -20\log_{10}(1 - \beta\lambda_{\min}) \\ &= \frac{-20\log_e(1 - \beta\lambda_{\min})}{\log_e 10} \\ &\approx \frac{20\beta\lambda_{\min}}{\log_e 10}.\end{aligned} \tag{S.490}$$

Where we used $\log(1 + x) \approx x$ for small x. As β increases, the speed of convergence increases in direct proportion.

Problem 9-9. The MSE is

$$E[Y_k - \hat{Y}_k]^2 = E[Y_k - aY_{k-1} - b]^2 \tag{S.491}$$

and setting the derivative w.r.t. a and b to zero,

$$0 = \phi_1 - a\phi_0 - b\mu = \mu - a\mu - b\ . \tag{S.492}$$

Solving for a and b we get the stated results.

Problem 9-10.

(a) Doing a partial fraction expansion, we get

$$\Phi(z) = \frac{A}{1-\alpha^2}\left[\frac{1}{1-\alpha z} + \frac{\alpha}{z-\alpha}\right] \tag{S.493}$$

and expanding each term, the first corresponding to positive time and the second to negative, we get

$$\phi_k = \frac{A}{1-\alpha^2}\alpha^{|k|}\ . \tag{S.494}$$

(b) Putting Y_k through a filter $(1 - \alpha z^{-1})$ results in a white signal. This filter is in the form of a predictor, and hence is the optimal predictor of infinite order. Hence the optimal predictor of any order one or higher has all-zero tap coefficients except for $f_1 = \alpha$.

Problem 9-11.

(a) From (9.38), we know that

$$\boldsymbol{q}_{j+1} = \sum_{i=1}^{n} (1 - \beta_{j+1}\lambda_i)(\boldsymbol{v}_i^* \boldsymbol{q}_j)\boldsymbol{v}_i. \tag{S.495}$$

Using this result and the assumption that the eigenvectors are orthonormal, we can prove the stated result by induction.

(b) We force the error to zero after N iterations by choosing

$$\beta_j = \frac{1}{\lambda_j}, \quad 1 \le j \le N. \tag{S.496}$$

The product term then always contains a term $1 - \beta_l \lambda_l = 0$ for every i.

Problem 9-12. Note: The problem should state that R_k is a real-valued and zero-mean process.

$$E[\sigma_k^2] = (1 - \alpha)\sum_{j=0}^{\infty} \alpha^j \sigma^2 = \sigma^2 \ . \tag{S.497}$$

(a) A key fact is that when X is a Gaussian zero-mean random variable, $E[X^3] = 0$ and $E[X^4] = 3\sigma^4$, which can be derived from the moment-generating function (Section 3.1). Then calculating the variance,

$$\mathrm{Var}[\sigma_k^2] = E[(\sigma_k^2)^2] - (\sigma^2)^2 \ . \tag{S.498}$$

The first term can be calculated directly,

$$(1 - \alpha)^2 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \alpha^{m+n} E[R_{k-m}^2 R_{k-n}^2]$$

$$= (1 - \alpha)^2 \left[\sum_{m=0}^{\infty} \alpha^{2m} E[R_{k-m}^4] + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \alpha^{m+n} (\sigma^2)^2 - \sum_{m=0}^{\infty} \alpha^{2m} (\sigma^2)^2 \right] \tag{S.499}$$

where the third term subtracts off terms that were included twice in the first and second terms. Using the result for the fourth moment, we get

$$Var[\sigma_k^2] = 2\sigma^4 \frac{1 - \alpha}{1 + \alpha}, \tag{S.500}$$

where this variance approaches zero as $\alpha \to 1$ and $2\sigma^4$ as $\alpha \to 0$. It is of course desirable to have an α near unity because of the long time constant, since this results in a lot of averaging. The price we pay is a long convergence time, or poor tracking capability.

Problem 9-13. In Fig. 8-18(c), let the samples at the first LPF output be R_k, so that this implies that

$$Y_k = \sum_{m=-N}^{N} c_m R_{2k-m} \quad \text{and} \quad E_k = A_k - Y_k \tag{S.501}$$

For simplicity, doing the real-valued case,

$$\frac{\partial}{\partial c_j} E_k{}^2 = 2E_k R_{2k-j} \ . \tag{S.502}$$

Then analogous to (9.54),

$$[\boldsymbol{c}_{k+}1]_j = [\boldsymbol{c}_k]_j - \beta E_k R_{2k-j} \ . \tag{S.503}$$

For each increment of k, the delay line storing R_k shifts two positions.

Problem 9-14.

(a) Φ is replaced by $(\Phi + \sigma^2\mathbf{I})$, and it is simple to verify that the eigenvalues of this matrix are $(\lambda_i + \sigma^2)$, with the same eigenvectors.

(b) The new eigenvalue spread is $(\lambda_{\max} + \sigma^2)/(\lambda_{\min} + \sigma^2)$, which is smaller than before.

(c) From Exercise 9-7,

$$(\Phi + \sigma^2\mathbf{I})^{-1} = \sum_{i=1}^{N} \frac{1}{\lambda_k + \sigma^2} \boldsymbol{v}_i \boldsymbol{v}_i{}^* \tag{S.504}$$

and hence

$$(\boldsymbol{c} - \Phi^{-1}\alpha) = \sum_{i=1}^{N} \frac{\sigma^2}{\lambda_i(\lambda_i + \sigma^2)} (\boldsymbol{v}_i{}^*\alpha)\boldsymbol{v}_i \tag{S.505}$$

and the excess MSE is

$$(\boldsymbol{c} - \Phi^{-1}\alpha)^*\Phi(\boldsymbol{c} - \Phi^{-1}\alpha) = \sigma^4 \sum_{i=1}^{N} \frac{|\boldsymbol{v}_i{}^*\alpha|^2}{\lambda_i(\lambda_l + \sigma^2)} \tag{S.506}$$

which is the same result as Problem 9-15. For small σ^2 the increase in MSE is approximately proportional to $(\sigma^2/\lambda_i)^2$ for the i-th mode.

Problem 9-15. Adding $\mu\ \|\boldsymbol{c}\|^2$ to (9.8),

$$E\left[E_k{}^2\right] = E[\,|A_k|^2] - 2\mathrm{Re}\{\boldsymbol{c}^*\alpha\} + \boldsymbol{c}^*(\Phi + \mu\mathbf{I})\boldsymbol{c} \ . \tag{S.507}$$

Hence the solution is the same as before with Φ replaced by $(\Phi + \mu\mathbf{I})$, or

$$\boldsymbol{c}_\mu = (\Phi + \mu\mathbf{I})^{-1}\alpha \ . \tag{S.508}$$

The eigenvalues of $(\Phi + \mu\mathbf{I})$ are $\boldsymbol{v}_i$ and the eigenvalues $(\lambda_i + \mu)$, where $\boldsymbol{v}_i$ and λ_i are the eigenvectors and eigenvalues of Φ. The eigenvectors and eigenvalues of $(\Phi + \mu\mathbf{I})^{-1}$ are $\boldsymbol{v}_i$ and $1/(\lambda_i + \mu)$ and hence

$$(\Phi + \mu\mathbf{I})^{-1} = \sum_{i=1}^{n} \frac{1}{\lambda_i + \mu} \boldsymbol{v}_i \boldsymbol{v}_i{}^* \tag{S.509}$$

and

$$\boldsymbol{c}_\mu - \boldsymbol{c}_{\mathrm{opt}} = \sum_{i=1}^{n} \left(\frac{1}{\lambda_i + \mu} - \frac{1}{\lambda_i}\right) \boldsymbol{v}_i \boldsymbol{v}_i{}^*\alpha \ . \tag{S.510}$$

The excess mse is given by (9.42),

$$E[E_k^2(T)] - E[E_k^2(T)]_{\min} = \sum_{i=1}^{n} \lambda_i[(\boldsymbol{c}_\mu - \boldsymbol{c}_{\text{opt}})^*\boldsymbol{v}_i]^2$$

$$= \sum_{i=1}^{n} \lambda_i \left[\frac{\mu \boldsymbol{v}_i^* \alpha}{\lambda_i(\lambda_i + \mu)} \right]^2 \tag{S.511}$$

The excess mse of the ith mode increases as

$$\left(\frac{\mu}{\lambda_i + \mu} \right)^2 \tag{S.512}$$

which has derivative

$$\frac{\partial}{\partial \mu} \left(\frac{\mu}{\lambda_i + \mu} \right)^2 = \frac{2\mu\lambda_i}{(\lambda_i + \mu)^3} \tag{S.513}$$

and since this derivative is zero at $\mu = 0$, the excess mse increases very slowly with μ.

Problem 9-16.

(a)

$$\nabla_{\boldsymbol{c}}\, \mu \|\boldsymbol{c}\|^2 = 2\mu\boldsymbol{c} \tag{S.514}$$

so that the gradient algorithm is

$$\boldsymbol{c}_{i+1} = (1 - \beta\mu)\boldsymbol{c}_i + \beta(\alpha - \boldsymbol{\Phi}\boldsymbol{c}_i). \tag{S.515}$$

(b) The eigenvalues of $(\boldsymbol{\Phi} - \mu\mathbf{I})$ are $(\lambda_i + \mu)$, so replace $\lambda_{\min}$ by $(\lambda_{\min} + \mu)$ and $\lambda_{\max}$ by $(\lambda_{\max} + \mu)$. The algorithm is stable if

$$0 < \beta < \frac{2}{\lambda_{\max} + \mu}. \tag{S.516}$$

(c)

$$\beta_{\text{opt}} = \frac{2}{\lambda_{\min} + \lambda_{\max} + 2\mu} \tag{S.517}$$

(d) The eigenvalue spread is now

$$\frac{\lambda_{\max} + \mu}{\lambda_{\min} + \mu} \approx \frac{\lambda_{\max}}{\lambda_{\max} + \mu}, \tag{S.518}$$

which can be reduced dramatically by μ.

(e) They apply to the speed of convergence of the average trajectory and the asymptotic excess mse caused by the algorithms converging to $\boldsymbol{c}_\mu$ rather than $\boldsymbol{c}_{\text{opt}}$. The second term in (S.518) biases the solution in the direction of keeping the coefficients small. The SG algorithm corresponding to minimizing (S.518) is

$$\boldsymbol{c}_{k+1} = (1 - \beta\mu)\boldsymbol{c}_k + \beta E_k \overline{\boldsymbol{p}}_k\ , \tag{S.519}$$

where $\mu = 0$ corresponds to the previous algorithm without leakage (9.53). The operation of this algorithm is evident, since the coefficient vector is multiplied by a constant slightly less than unity at each step before adding in the correction term. When the corrections are small, as when the coefficient vector is wandering in the direction of a eigenvector corresponding to a small eigenvalue, this leakage decreases the size of the vector over time.

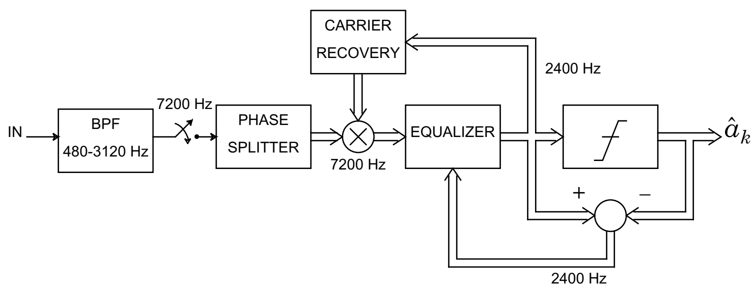
Problem 9-17. The lowest frequency in the passband signal is

$$1800 - 1200 \cdot 1.1 = 480 \text{ Hz} \tag{S.520}$$

and the highest frequency is

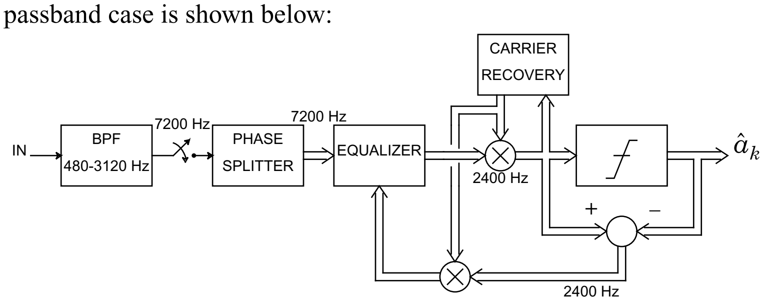
$$1800 + 1200 \cdot 1.1 = 3120 \text{ Hz} \ . \tag{S.521}$$

(a) The baseband case is shown below:



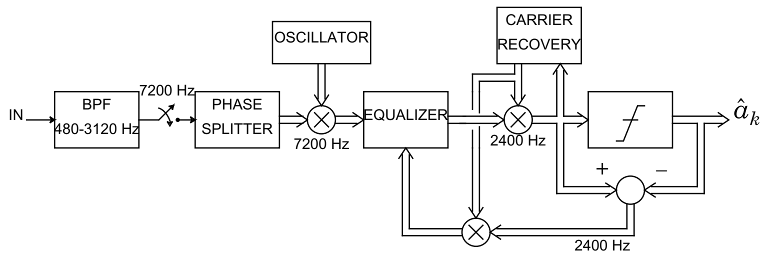
The BPF rejects all frequencies other than the signal bandwidth. The sampling rate at the front end of 7200 Hz is greater than twice the highest frequency of the passband signal. Following demodulation, a sampling rate of 4800 Hz would be adequate since the highest signal frequency is 1320 Hz; however, it is not convenient to decimate by a factor of two-thirds (perhaps a 9600-4800 decimation would be more appropriate, but this does not meet the specifications of the problem statement). The fractionally spaced equalizer can generate a signal at the slicer input at the symbol rate, a decimation factor of three.

(b) The passband case is shown below:



You might expect that the equalizer output had to be sampled at 7200 Hz also, but after demodulation a rate of 2400 is adequate. Since demodulation followed by decimation is equivalent to decimation followed by demodulation, in fact the equalizer output can be decimated to the symbol rate.

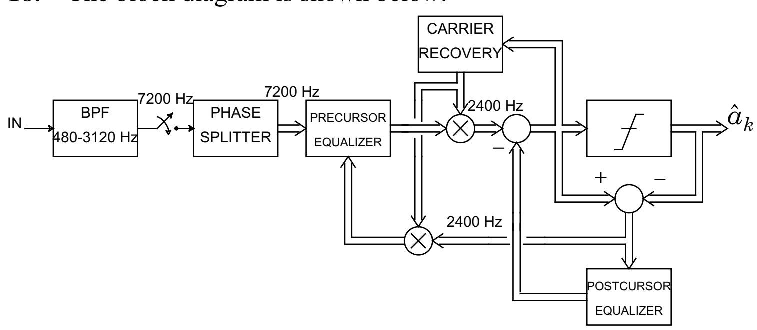
(c) This case is shown below:



This case is similar, except for the reasoning. The equalizer input could again have a sampling rate of 4800, although this is not convenient. The 2400 Hz sampling rate at equalizer output is adequate for the same reason as in the baseband case (no funny business as in the passband case).

The passband equalizer case seems superior since the sampling rates are the same, but only one complex multiply for demodulation is required rather than two.

Problem 9-18. The block diagram is shown below:



The postcursor equalizer and carrier recovery circuits work in the non-rotated domain along with the slicer. The slicer error is used directly in the postcursor equalizer and is rotated before being input to the precursor equalizer.

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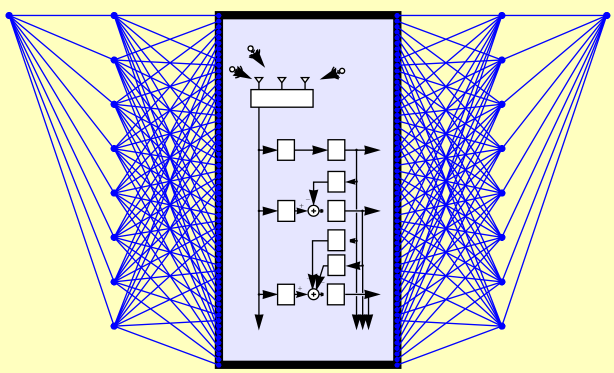

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Chapter 10

Problem 10-1. The $m \times m$ power spectrum to be factored is:

$$\begin{aligned}
\mathbf{S}(z) &= \begin{bmatrix} 64.01 & 1.6z + 8 + 0.2z^{-1} \\ 0.2z + 8 + 1.6z^{-1} & 0.2z + 5.04 + 0.2z^{-1} \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1.6 \\ 0.2 & 0.2 \end{bmatrix} z + \begin{bmatrix} 64.01 & 8 \\ 8 & 5.04 \end{bmatrix} + \begin{bmatrix} 0 & 0.2 \\ 1.6 & 0.2 \end{bmatrix} z^{-1} \\
&= \mathbf{S}_{-1} z + \mathbf{S}_0 + \mathbf{S}_1 z^{-1} ,
\end{aligned} \tag{S.523}$$

where

$$\mathbf{S}_{-1} = \begin{bmatrix} 0 & 1.6 \\ 0.2 & 0.2 \end{bmatrix}, \qquad \mathbf{S}_0 = \begin{bmatrix} 64.01 & 8 \\ 8 & 5.04 \end{bmatrix}, \qquad \mathbf{S}_1 = \begin{bmatrix} 0 & 0.2 \\ 1.6 & 0.2 \end{bmatrix}. \tag{S.524}$$

(a) With $k = 1$ and $m = 2$, the $m(k + 1) \times m(k + 1)$ matrix $\mathbf{R}_k$ reduces to:

$$\mathbf{R}_1 = \begin{bmatrix} \mathbf{S}_0 & \mathbf{S}_{-1} \\ \mathbf{S}_1 & \mathbf{S}_0 \end{bmatrix} = \begin{bmatrix} 64.01 & 8 & 0 & 1.6 \\ 8 & 5.04 & 0.2 & 0.2 \\ 0 & 1.2 & 64.01 & 8 \\ 1.6 & 0.2 & 8 & 5.04 \end{bmatrix}. \tag{S.525}$$

A Cholesky factorization of this 4×4 matrix yields:

$$\mathbf{R}_1 = \mathbf{L}_1 \mathbf{L}_1^T \tag{S.526}$$

where (for example, using `L1 = chol(R1)'` in MATLAB):

$$\mathbf{L}_1 = \begin{bmatrix} 8.0006 & 0 & 0 & 0 \\ 8 & 2.0100 & 0 & 0 \\ 0 & 0.0995 & 8.0000 & 0 \\ 0.2000 & 0.0000 & 1.0000 & 2.0000 \end{bmatrix}. \tag{S.527}$$

Equating the last *block* row (the last two rows) of this matrix with = $[\mathbf{G}_1, \mathbf{G}_0]$ yields:

$$\mathbf{G}_0 = \begin{bmatrix} 8.0000 & 0 \\ 1.0000 & 2.0000 \end{bmatrix}, \qquad \mathbf{G}_1 = \begin{bmatrix} 0 & 0.0995 \\ 0.2000 & 0.0000 \end{bmatrix}. \tag{S.528}$$

Thus, the Bauer method with $k = 1$ leads to the following approximation:

$$\mathbf{G}(z) \approx \begin{bmatrix} 8.0000 & 0.0995z^{-1} \\ 1.0000 + 0.20000z^{-1} & 2.0000 + 0.0000z^{-1} \end{bmatrix}. \tag{S.529}$$

(b) The approximation from part (a) is not exact, because it gives:

$$
\begin{aligned}
\mathbf{G}(z)\mathbf{G}^*(1/z^*) &= (\mathbf{G}_0 + \mathbf{G}_1 z^{-1})(\mathbf{G}_0{}^T + \mathbf{G}_1{}^T z) \\
&= \mathbf{G}_0\mathbf{G}_1{}^T z + (\mathbf{G}_0\mathbf{G}_0{}^T + \mathbf{G}_1\mathbf{G}_1{}^T) + \mathbf{G}_1\mathbf{G}_0{}^T z^{-1} \\
&= \begin{bmatrix} 0 & 1.5999 \\ 0.1990 & 0.2000 \end{bmatrix} z + \begin{bmatrix} 64.010 & 8.0000 \\ 8.0000 & 5.0400 \end{bmatrix} + \begin{bmatrix} 0 & 0.1990 \\ 1.5999 & 0.2000 \end{bmatrix} z^{-1}. \qquad \text{(S.530)}
\end{aligned}
$$

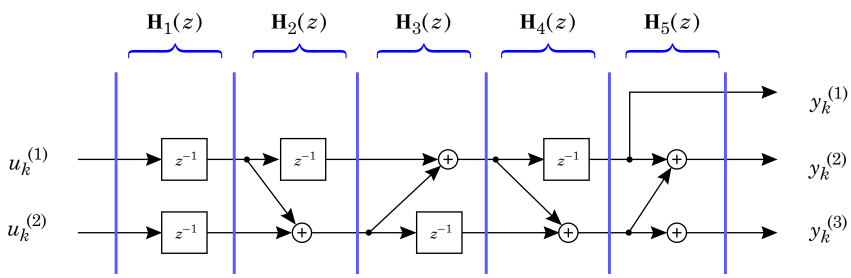
However, comparing this to the original $\mathbf{S}(z)$ of (S.523), the error is small:

$$
\mathbf{G}(z)\mathbf{G}^*(1/z^*) - \mathbf{S}(z) = \begin{bmatrix} 0 & -0.00001237z - 0.00009962z^{-1} \\ -0.00009972z - 0.00001237z^{-1} & 0.00000153(z + z^{-1}) \end{bmatrix}, \qquad \text{(S.531)}
$$

which suggests that the $\mathbf{G}(z)$ of (S.528) is nearly correct, and that by changing the 0.0995 in $\mathbf{G}_1$ to 0.1 might lead to the exact answer. In fact this is true, the actual factor (as is easily verified) is:

$$
\mathbf{G}(z) = \begin{bmatrix} 8 & 0 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 0.1 \\ 0.2 & 0 \end{bmatrix} z^{-1}. \qquad \text{(S.532)}
$$

Problem 10-2. The overall filter can be decomposed into a cascade of five filters:



The transfer functions of each are easily found by inspection, so that:

$$
\begin{aligned}
\mathbf{H}(z) &= \mathbf{H}_5(z)\mathbf{H}_4(z)\mathbf{H}_3(z)\mathbf{H}_2(z)\mathbf{H}_1(z) \\
&= \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z^{-1} & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & z^{-1} \end{bmatrix} \begin{bmatrix} z^{-1} & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-1} \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z^{-1} & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & z^{-1} \end{bmatrix} \begin{bmatrix} z^{-2} & 0 \\ z^{-1} & z^{-1} \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z^{-1} & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} z^{-1} + z^{-2} & z^{-1} \\ z^{-2} & z^{-2} \end{bmatrix}
\end{aligned}
$$

$$= \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z^{-2}+z^{-3} & z^{-2} \\ z^{-1}+z^{-2}+z^{-3} & z^{-1}+z^{-2} \end{bmatrix}$$

$$= \begin{bmatrix} z^{-2}+z^{-3} & z^{-2} \\ z^{-1}+2z^{-2}+2z^{-3} & z^{-1}+2z^{-2} \\ z^{-1}+z^{-2}+z^{-3} & z^{-1}+z^{-2} \end{bmatrix}. \tag{S.533}$$

Problem 10-3. The relationship between $\boldsymbol{x}_k = [y_k^{(1)}, y_k^{(3)}]$ and $\boldsymbol{y}_k = [y_k^{(1)}, y_k^{(2)}, y_k^{(3)}]$ is:

$$\boldsymbol{x}_k = \mathbf{Q}\boldsymbol{y}_k, \qquad \text{where } \mathbf{Q} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{S.534}$$

and therefore, the transfer function from $\boldsymbol{u}_k = [u_k^{(1)}, u_k^{(2)}]$ to $\boldsymbol{x}_k$ is:

$$\begin{aligned} \tilde{\mathbf{H}}(z) &= \mathbf{Q}\mathbf{H}(z) \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z^{-2}+z^{-3} & z^{-2} \\ z^{-1}+2z^{-2}+2z^{-3} & z^{-1}+2z^{-2} \\ z^{-1}+z^{-2}+z^{-3} & z^{-1}+z^{-2} \end{bmatrix} \\ &= \begin{bmatrix} z^{-2}+z^{-3} & z^{-2} \\ z^{-1}+z^{-2}+z^{-3} & z^{-1}+z^{-2} \end{bmatrix} \\ &= z^{-1} \begin{bmatrix} z^{-1}(1+z^{-1}) & z^{-1} \\ 1+z^{-1}+z^{-2} & (1+z^{-1}) \end{bmatrix} \end{aligned} \tag{S.535}$$

Therefore, exploiting the fact that $\mathbf{S}_{\boldsymbol{u}}(z) = \mathbf{I}$ yields:

$$\begin{aligned} \mathbf{S}_{\boldsymbol{x}}(z) &= \tilde{\mathbf{H}}(z)\mathbf{S}_{\boldsymbol{u}}(z)\tilde{\mathbf{H}}^*(1/z^*) \\ &= \tilde{\mathbf{H}}(z)\tilde{\mathbf{H}}^*(1/z^*) \\ &= z^{-1}z \begin{bmatrix} z^{-1}(1+z^{-1}) & z^{-1} \\ 1+z^{-1}+z^{-2} & (1+z^{-1}) \end{bmatrix} \begin{bmatrix} z(1+z) & 1+z+z^2 \\ z & (1+z) \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} (z^{-1}(1+z^{-1}))(z(1+z)) + (z^{-1})(z) & (z^{-1}(1+z^{-1}))(1+z+z^2) + (z^{-1})(1+z) \\ (1+z^{-1}+z^{-2})(z(1+z)) + (1+z^{-1})(z) & (1+z^{-1}+z^{-2})(1+z+z^2) + (1+z^{-1})(1+z) \end{bmatrix}$$

$$= \begin{bmatrix} z+3+z^{-1} & z+3+3z^{-1}+z^{-2} \\ z^2+3z+3+z^{-1} & z^{-2}+3z^{-1}+5+3z+z^2 \end{bmatrix}$$

$$\mathbf{S}_{\boldsymbol{x}}(z) = \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}}_{\mathbf{R}_{\boldsymbol{x}}(-2)} z^2 + \underbrace{\begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}}_{\mathbf{R}_{\boldsymbol{x}}(-1)} z + \underbrace{\begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix}}_{\mathbf{R}_{\boldsymbol{x}}(0)} + \underbrace{\begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix}}_{\mathbf{R}_{\boldsymbol{x}}(1)} z^{-1} + \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}}_{\mathbf{R}_{\boldsymbol{x}}(2)} z^{-2}. \tag{S.536}$$

(a) The autocorrelation function $\mathbf{R}_{\boldsymbol{x}}(m)$ at lag m is the coefficient of z^{-m} in (S.536).

(b) The power spectrum $\mathbf{S}_{\boldsymbol{x}}(z)$ is given by (S.536).

Problem 10-4. The filter input power spectrum is a constant, and it can be factored as:

$$\mathbf{S}_{\boldsymbol{X}}(z) = \begin{bmatrix} 16 & 8 & 4 \\ 8 & 20 & 10 \\ 20 & 10 & 21 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/4 & 1/2 & 1 \end{bmatrix}}_{\mathbf{M}} \underbrace{\begin{bmatrix} 16 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 16 \end{bmatrix}}_{\Gamma^2} \underbrace{\begin{bmatrix} 1 & 1/2 & 1/4 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{M}^*} . \tag{S.537}$$

(a) We want the output power spectrum to be $\mathbf{S}(z) = \mathbf{G}(z)\mathbf{G}^*(1/z^*)$, where from (10.18):

$$\mathbf{G}(z) = \begin{bmatrix} 10 & 0 \\ 1 & 20 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} z^{-1}. \tag{S.538}$$

This can be achieved by first creating an *intermediate* signal $\boldsymbol{U}_k = \mathbf{A}\boldsymbol{X}_k$, with $\mathbf{A}$ chosen so that $\boldsymbol{U}_k$ is both temporally and spatially white, satisfying $\mathbf{S}_{\boldsymbol{U}}(z) = \mathbf{I}$. In other words, we choose $\mathbf{A}$ so that:

$$\mathbf{A}\mathbf{S}_{\boldsymbol{X}}(z)\mathbf{A}^* = \mathbf{I} \, . \tag{S.539}$$

In light of (S.537), one obvious choice for $\mathbf{A}$ is:

$$\mathbf{A} = (\mathbf{M}\Gamma)^{-1} = \begin{bmatrix} 4 & 0 & 0 \\ 2 & 4 & 0 \\ 1 & 2 & 4 \end{bmatrix}^{-1} = \frac{1}{8}\begin{bmatrix} 2 & 0 & 0 \\ -1 & 2 & 0 \\ 0 & -1 & 2 \end{bmatrix} . \tag{S.540}$$

In light of (10.11), passing $\boldsymbol{U}_k$ through the following 3-input 2-output filter:

$$\mathbf{F}(z) = \begin{bmatrix} \mathbf{G}(z) & \begin{matrix} 0 \\ 0 \end{matrix} \end{bmatrix} \tag{S.541}$$

will result in the desired power spectrum at the output. Overall, the desired filter is the cascade of $\mathbf{A}$ and $\mathbf{F}(z)$:

$$\begin{aligned} \mathbf{H}(z) &= \mathbf{F}(z)\mathbf{A} \\ &= \frac{1}{8}\begin{bmatrix} 10 + z^{-1} & 2z^{-1} & 0 \\ 1 + 3z^{-1} & 20 + 4z^{-1} & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ -1 & 2 & 0 \\ 0 & -1 & 2 \end{bmatrix} \\ &= \frac{1}{4}\begin{bmatrix} 10 & 2z^{-1} & 0 \\ -9 + z^{-1} & 20 + 4z^{-1} & 0 \end{bmatrix} . \end{aligned} \tag{S.542}$$

(b) The answer to part (a) is not unique. The filter of (S.541) drops the third component of $\mathbf{U}_k$, but this is arbitrary; we could also have dropped the first or second component. Also, the whitening filter $\mathbf{A}$ is not unique; any $\mathbf{A}$ of the form $(\mathbf{M}\Gamma)^{-1}\Theta$ where Θ is a unitary matrix will give the same answer.

Problem 10-5. Let us solve the problem for the general set of pulses $h_1(t) = e^{-t/\tau}u(t-T)$ and $h_2(t) = e^{-t/\tau}u(t-2T)$. That way we can handle both cases (τ finite and $\tau \to \infty$) simultaneously.

It is not hard to see that, for any SIMO channel, the folded spectrum collapses to the sum of the folded spectra for each of the subchannels. In particular, whenever $\mathbf{H}(t)$ is of the form:

$$\boldsymbol{H}(t) = \begin{bmatrix} h_1(t) \\ h_2(t) \\ \vdots \\ h_n(t) \end{bmatrix} \quad \leftrightarrow \quad \boldsymbol{H}(f) = \begin{bmatrix} H_1(f) \\ H_2(f) \\ \vdots \\ H_n(f) \end{bmatrix} \tag{S.543}$$

then the folded spectrum is, by definition (10.40):

$$\begin{aligned} \mathbf{S}(e^{j2\pi fT}) &= \tfrac{1}{T}\textstyle\sum_k \mathbf{H}^*(f-\tfrac{k}{T})\mathbf{H}(f-\tfrac{k}{T}) \\ &= \tfrac{1}{T}\textstyle\sum_k \begin{bmatrix} H_1^*(f-k/T) & H_2^*(f-k/T) & \ldots & H_n^*(f-k/T) \end{bmatrix} \begin{bmatrix} H_1(f-k/T) \\ H_2(f-k/T) \\ \vdots \\ H_n(f-k/T) \end{bmatrix} \\ &= \tfrac{1}{T}\textstyle\sum_k \left| H_1\left(f-\tfrac{k}{T}\right)\right|^2 + \ldots + \tfrac{1}{T}\sum_k \left| H_1\left(f-\tfrac{k}{T}\right)\right|^2 \\ &= S_{h_1}(f) + S_{h_2}(f) + \ldots + S_{h_n}(f)\,. \end{aligned} \tag{S.544}$$

This implies that, in the Z domain, the SIMO folded spectrum is $S(z) = \sum_i S_{h_i}(z)$, where $S_{h_i}(z)$ is the folded spectrum of the i-th pulse.

In this problem we have $h_1(t) = e^{-t/\tau}u(t-T)$, so that the first sampled autocorrelation function evaluated at $k \ge 0$ is:

$$\begin{aligned} \rho_{h_1}(k) &= \int_{-\infty}^{\infty} h_1(t)h_1(t-kT)dt \\ &= \int_{-\infty}^{\infty} h_1(t+T)h_1(t+T-kT)dt \\ &= e^{-2T/\tau}\int_{kT}^{\infty} e^{-t/\tau}e^{-(t+kT)/\tau}dt \qquad \text{(assuming } k \ge 0\text{)} \\ &= e^{-(k+2)T/\tau}\int_0^{\infty} e^{-2t/\tau}dt \qquad \text{(assuming } k \ge 0\text{)} \\ &= E_1 a^k\,, \qquad \text{(assuming } k \ge 0\text{)} \end{aligned} \tag{S.545}$$

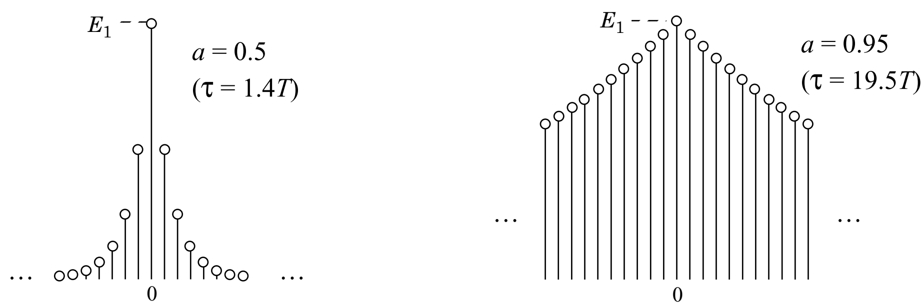
where $a = e^{-T/\tau}$, and where E_1 is the energy in the first pulse:

$$E_1 = \int_{-\infty}^{\infty} |\,h_1(t)\,|^2 dt = \frac{\tau}{2}a^2\,. \tag{S.546}$$

Exploiting the Hermitian symmetry $\rho_h(-k) = \rho_h^*(k)$, we immediately conclude that:

$$\rho_{h_1}(k) = E_1 a^{|k|} \ . \tag{S.547}$$

Here are two sketches of $\rho_{h_1}(k)$, one for small τ / T and one for large:



We can decompose $\rho_{h_1}(k)$ into the sum $\rho_{h_1}(k) = x_k + y_k$, where $x_k = E_1 a^k u_k$ is causal and y_k is strictly anticausal:

$$\begin{aligned} y_k &= E_1 a^{-k} u_{-k-1} \\ &= a E_1 a^{-k-1} u_{-k-1} \\ &= a x_{-k-1} \ . \end{aligned} \tag{S.548}$$

The Z transform of $x_k = E_1 a^k u_k$ is $E_1 / (1 - az^{-1})$. From the properties of Z transforms, we know that the Z transform of x_{k-1} is $z^{-1} X(z)$, and that the Z transform of $h^*_{-k} = H^*(1/z^*)$. Setting $h_k = x_{k-1}$ implies that the Z transform of $y_k = a h_{-k}$ is $Y(z) = aH^*(1/z^*) = azX^*(1/z^*)$. Combining these results, we see that the Z transform of $\rho_{h_1}(k)$ is:

$$\begin{aligned} S_{h_1}(z) &= X(z) + Y(z) \\ &= X(z) + azX^*(1/z^*) \\ &= \frac{E_1}{1 - az^{-1}} + \frac{E_1 az}{1 - az} \\ &= \frac{(1 - a^2)E_1}{(1 - az^{-1})(1 - az)} \ . \end{aligned} \tag{S.549}$$

Since $h_2(t) = a h_1(t - T)$, where again $a = e^{-T/\tau}$, it immediately follows that:

$$\rho_{h_2}(k) = a^2 \rho_{h_1}(k) \ , \tag{S.550}$$

so that $S_{h_2}(z) = a^2 S_{h_1}(z)$. The overall folded spectrum for the SIMO channel is thus:

$$\begin{aligned} S(z) &= S_{h_1}(z) + S_{h_2}(z) \\ &= S_{h_1}(z)(1 + a^2) \\ &= \frac{(1 - a^4)E_1}{(1 - az^{-1})(1 - az)} \ . \end{aligned} \tag{S.551}$$

Setting $\tau = 1$ leads to $a = e^{-T}$ and $E_1 = \frac{1}{2}e^{-2T}$, so that:

$$S(z) = \frac{(1/2)(1-e^{-4T})e^{-2T}}{(1-e^{-T}z^{-1})(1-e^{-T}z)} \qquad (\tau = 1)\ . \tag{S.552}$$

Letting $\tau \to \infty$ leads to $a \to 1$, so that:

$$S(z) = \frac{(1-a^4)E_1}{(1-az^{-1})(1-az)} \to \frac{A}{(1-z^{-1})(1-z)}\ , \tag{S.553}$$

where, by introducing $x = T/\tau$, so that $a = e^{-x}$ and $E_1 = \frac{T}{2x}e^{-2x}$, the numerator is:

$$\begin{aligned} A &= (1-a^4)E_1 \\ &= T\frac{1-e^{-4x}}{2xe^{2x}} \\ &= T\frac{4e^{-4x}}{4xe^{2x}+2e^{2x}} \to 2T \text{ as } \tau \to \infty\ (x \to 0)\ , \end{aligned} \tag{S.554}$$

where the last equality follows from L'Hopital's rule. This leads to the following limit:

$$S(z) \to \frac{2T}{(1-z^{-1})(1-z)} \quad \text{as} \quad \tau \to \infty. \tag{S.555}$$

Problem 10-6. There is a typo in the problem statement, the problem cannot be solved as stated. The folded spectrum of a *SIMO* channel is always a scalar, there is no way we can force it to have dimension 2×2.

We will solve the intended problem, where "single-input double output" should be "double-input single-output". From (10.38), the folded spectrum is then $\mathbf{S}(z) = \sum_k \mathbf{S}_k z^{-k}$, where

$$\mathbf{S}_k = \int_{-\infty}^{\infty} \mathbf{H}^*(t-kT)\mathbf{H}(t)\,dt\ . \tag{S.556}$$

For a double-input single-output channel with $\mathbf{H}(t) = [h_1(t),\ h_2(t)]$, this reduces to:

$$\mathbf{S}_k = \int_{-\infty}^{\infty} \begin{bmatrix} h_1^*(t-kT) \\ h_2^*(t-kT) \end{bmatrix} \begin{bmatrix} h_1(t),\ h_2(t) \end{bmatrix} dt \tag{S.557}$$

$$= \begin{bmatrix} \int_{-\infty}^{\infty} h_1^*(t-kT)h_1(t)\,dt & \int_{-\infty}^{\infty} h_1^*(t-kT)h_2(t)\,dt \\ \int_{-\infty}^{\infty} h_2^*(t-kT)h_1(t)\,dt & \int_{-\infty}^{\infty} h_2^*(t-kT)h_2(t)\,dt \end{bmatrix} \tag{S.558}$$

Setting $k = 0$ leads to $\mathbf{S}_0$, and equating it to the coefficient of z^0 in the target spectrum yields:

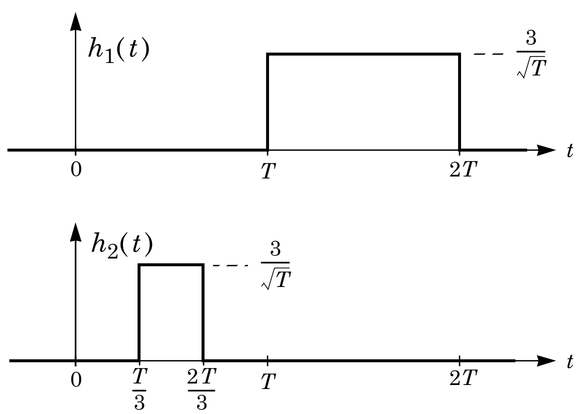
$$\mathbf{S}_0 = \begin{bmatrix} \int_{-\infty}^{\infty} |h_1(t)|^2 dt & \int_{-\infty}^{\infty} h_1^*(t) h_2(t) dt \\ \int_{-\infty}^{\infty} h_2^*(t) h_1(t) dt & \int_{-\infty}^{\infty} |h_2(t)|^2 dt \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 3 \end{bmatrix}. \tag{S.559}$$

This implies that the first pulse has energy 9, the second has energy 3, and the two pulses are orthogonal.

Setting $k = 1$ leads to $\mathbf{S}_1$, and equating it to the coefficient of z^{-1} in the target spectrum yields:

$$\mathbf{S}_1 = \begin{bmatrix} \int_{-\infty}^{\infty} h_1^*(t-T) h_1(t) dt & \int_{-\infty}^{\infty} h_1^*(t-T) h_2(t) dt \\ \int_{-\infty}^{\infty} h_2^*(t-T) h_1(t) dt & \int_{-\infty}^{\infty} h_2^*(t-T) h_2(t) dt \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix}. \tag{S.560}$$

The diagonal elements imply that both pulses are orthogonal to delayed-by-T versions of themselves. The following pair of impulse responses gives the desired power spectrum:



Clearly the first pulse has energy 9, the second has energy 3, and the two pulses are orthogonal, so (S.559) is satisfied. Further, the sampled crosscorrelation function:

$$[\mathbf{S}_k]_{2,1} = \int_{-\infty}^{\infty} h_2{}^*(t - kT) h_1(t) dt \tag{S.561}$$

is nonzero only when $k = 1$, in which case is reduces to $[\mathbf{S}_1]_{2,1} = 3$. So (S.560) is satisfied.

Problem 10-7. The proposition is true, it is a generalized version of Parseval's relationship applied to MIMO filters. The proof is based on the inverse Fourier transform equation:

$$\mathbf{B}_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathbf{B}(e^{j\theta}) e^{jk\theta} \, d\theta \ . \tag{S.562}$$

Making this substitution, the right-hand-side of the proposition reduces to:

$$\sum_{k=-\infty}^{\infty}\mathbf{A}_k\mathbf{B}_k^* = \sum_{k=-\infty}^{\infty}\mathbf{A}_k\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathbf{B}^*(e^{j\theta})e^{-jk\theta}\,\mathrm{d}\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left\{\sum_{k=-\infty}^{\infty}\mathbf{A}_k e^{-jk\theta}\right\}\mathbf{B}^*(e^{j\theta})\,\mathrm{d}\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left\{\mathbf{A}(e^{j\theta})\right\}\mathbf{B}^*(e^{j\theta})\,\mathrm{d}\theta\,, \qquad \text{(S.563)}$$

which is the left-hand side of the proposition.

Problem 10-8. In terms of $\boldsymbol{x}_k = [x_k^{(1)}, x_k^{(2)}]^T$, the given linear predictor can be written as:

$$\hat{x}_k^{(1)} = p_1 x_{k-1}^{(1)} + p_2 x_{k-1}^{(2)}$$
$$= \mathbf{p}^T\boldsymbol{x}_{k-1}\,, \qquad \text{(S.564)}$$

where $\mathbf{p} = [p_1, p_2]^T$. The quantity to be predicted is $x_k^{(1)}$, the first component of $\boldsymbol{x}_k$:

$$x_k^{(1)} = \mathbf{e}^T\boldsymbol{x}_k\ , \qquad \text{(S.565)}$$

where $\mathbf{e} = [1, 0]^T$. The mean-squared prediction error is thus:

$$\begin{aligned}\mathrm{MSE} &= \mathrm{E}[(\hat{x}_k^{(1)} - x_k^{(1)})^2]\\ &= \mathrm{E}[(\mathbf{p}^T\boldsymbol{x}_{k-1} - \mathbf{e}^T\boldsymbol{x}_k)^2]\\ &= \mathbf{p}^T\mathrm{E}[\boldsymbol{x}_{k-1}\boldsymbol{x}_{k-1}^T]\mathbf{p} + \mathbf{e}^T\mathrm{E}[\boldsymbol{x}_k\boldsymbol{x}_k^T]\mathbf{e} - 2\mathbf{p}^T\mathrm{E}[\boldsymbol{x}_{k-1}\boldsymbol{x}_k^T]\mathbf{e}\\ &= \mathbf{p}^T\mathbf{R}_{\boldsymbol{x}}(0)\mathbf{p} + \mathbf{e}^T\mathbf{R}_{\boldsymbol{x}}(0)\mathbf{e} - 2\mathbf{p}^T\mathbf{R}_{\boldsymbol{x}}(-1)\mathbf{e}\,. \end{aligned} \qquad \text{(S.566)}$$

(We used $\mathbf{R}_{\boldsymbol{x}}(d) = \mathrm{E}[\boldsymbol{x}_{k+d}\boldsymbol{x}_k^*]$ from (10.6) or (10.9).)
Taking the gradient with respect to $\mathbf{p}$ yields (see (9.21) and (9.22)):

$$\nabla_{\mathbf{p}}\mathrm{MSE} = 2\mathbf{R}_{\boldsymbol{x}}(0)\mathbf{p} - 2\mathbf{R}_{\boldsymbol{x}}(-1)\mathbf{e}\,. \qquad \text{(S.567)}$$

The optimal (MMSE) prediction coefficients can be found by setting the gradient to zero:

$$\mathbf{p} = \mathbf{R}_{\boldsymbol{x}}(0)^{-1}\mathbf{R}_{\boldsymbol{x}}(-1)\mathbf{e}\,. \qquad \text{(S.568)}$$

Since $\mathbf{R}(d)$ is the coefficient of z^{-d} in $\mathbf{S}(z)$, the PSD given in (10.17) implies that:

$$\mathbf{R}_{\boldsymbol{x}}(0) = \begin{bmatrix}105 & 21\\ 21 & 426\end{bmatrix}, \qquad \mathbf{R}_{\boldsymbol{x}}(-1) = \begin{bmatrix}10 & 30\\ 41 & 83\end{bmatrix}. \qquad \text{(S.569)}$$

With these values, (S.568) reduces to:

$$\mathbf{p} = \begin{bmatrix}p_1\\ p_2\end{bmatrix} = \begin{bmatrix}0.0767\\ 0.0925\end{bmatrix}. \qquad \text{(S.570)}$$

It is instructive to compare the above “finite-tap” solution (where the number of prediction coefficients is severely constrained) to the “infinite-tap” unrestricted linear predictor of (10.26). The unconstrained solution is given by (10.29), namely:

$$\mathbf{P}(z) = \mathbf{I} - \mathbf{M}(z)^{-1}, \tag{S.571}$$

where from (10.18) $\mathbf{M}(z)$ satisfies:

$$\mathbf{M}(z)\Gamma = \begin{bmatrix} 10 & 0 \\ 1 & 20 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} z^{-1} \quad \Rightarrow \quad \Gamma = \begin{bmatrix} 10 & 0 \\ 0 & 20 \end{bmatrix}, \tag{S.572}$$

so that:

$$\mathbf{M}(z) = \begin{bmatrix} 1 + 0.1z^{-1} & 0.1z^{-1} \\ 0.1 + 0.3z^{-1} & 1 + 0.2z^{-1} \end{bmatrix}. \tag{S.573}$$

Therefore, in terms of the determinant $D(z) = \det\mathbf{M}(z)$, the infinite-tap predictor is:

$$\begin{aligned} \mathbf{P}(z) &= \mathbf{I} - \mathbf{M}(z)^{-1} \\ &= \frac{1}{D(z)}\begin{bmatrix} D(z) & 0 \\ 0 & D(z) \end{bmatrix} - \frac{1}{D(z)}\begin{bmatrix} 1 + 0.2z^{-1} & -0.1z^{-1} \\ -0.1 - 0.3z^{-1} & 1 + 0.1z^{-1} \end{bmatrix} \\ &= \frac{1}{D(z)}\begin{bmatrix} D(z) - 1 - 0.2z^{-1} & 0.1z^{-1} \\ 0.1 + 0.3z^{-1} & D(z) - 1 - 0.1z^{-1} \end{bmatrix} \\ &= \frac{1}{D(z)}\begin{bmatrix} 0.09z^{-1} - 0.01z^{-2} & 0.1z^{-1} \\ 0.1 + 0.3z^{-1} & 0.19z^{-1} - 0.01z^{-2} \end{bmatrix}, \end{aligned} \tag{S.574}$$

where we exploited the fact that:

$$D(z) = \det\mathbf{M}(z) = 1 + 0.29z^{-1} - 0.01z^{-2}\,. \tag{S.575}$$

Since $1/D(z)$ is a causal and monic filter, the first two prediction-filter coefficients are:

$$\mathbf{P}_0 = \begin{bmatrix} 0 & 0 \\ 0.1 & 0 \end{bmatrix}, \qquad \mathbf{P}_1 = \begin{bmatrix} 0.09 & 0.1 \\ \times & 0.19 \end{bmatrix}. \tag{S.576}$$

(The symbol $\times$ represents “don’t care”; it cannot be found by inspection, which would require knowledge of the inverse transform of $1/D(z)$, but for our purposes its value doesn’t matter.)

We are now ready to compare: the finite-tap predictor in this problem forces all predictor coefficients to be zero, except for the *first row* of $\mathbf{P}_1$. Comparing the first row of $\mathbf{P}_1$ to the finite-tap solution, we see that the finite-tap solution is very nearly (but not exactly) equal to the infinite-tap solution with the nonexistant taps are set to zero:

$$\mathrm{p} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 0.0767 \\ 0.0925 \end{bmatrix} \quad \approx \quad \{\text{first row of } \mathbf{P}_1\}^T = \begin{bmatrix} 0.09 \\ 0.1 \end{bmatrix}. \tag{S.577}$$

Problem 10-9. The JML detector chooses $(a_1, a_2)^T$ to maximize the conditional pdf $f(\boldsymbol{r} \mid a_1, a_2)$. It thus decides $a_1 = 1$ when

$$\max\{f(\boldsymbol{r} \mid 1, 1), f(\boldsymbol{r} \mid 1, -1)\} > \max\{f(\boldsymbol{r} \mid -1, 1), f(\boldsymbol{r} \mid -1, -1)\} \tag{S.578}$$

or equivalently, since the logarithm is monotonic:

$$\hat{a}_1^{JML} = \text{sign}\left\{\log\frac{\max\{f(\boldsymbol{r}|1,1), f(\boldsymbol{r}|1,-1)\}}{\max\{f(\boldsymbol{r}|-1,1), f(\boldsymbol{r}|-1,-1)\}}\right\}. \tag{S.579}$$

In contrast, the IML detector chooses its decision for a_1 to maximize

$$f(\boldsymbol{r}\,|\,a_1) = \frac{f(r|a_1, 1) + f(r|a_1, -1)}{2}. \tag{S.580}$$

In other words, the IML detector will choose $\hat{a}_1$ according to the sign of $\log\frac{f(\boldsymbol{r}|a_1 = 1)}{f(\boldsymbol{r}|a_1 = -1)}$:

$$\hat{a}_1^{IML} = \text{sign}\left\{\log\frac{f(r|1,1) + f(r|1,-1)}{f(r|-1,1) + f(r|-1,-1)}\right\}. \tag{S.581}$$

Comparing, we see that the IML and JML detectors can be put into the same form:

$$\hat{a}_1 = \text{sign}\left\{\log\frac{h\{f(r|1,1), f(r|1,-1)\}}{h\{f(r|-1,1), f(r|-1,-1)\}}\right\}, \tag{S.582}$$

where $h_{JML}(x, y) = \max(x, y)$ for the JML detector, and $h_{IML}(x, y) = x + y$ for the IML detector. But

$$\begin{aligned} f(\boldsymbol{r}\,|\,a_1, a_2) &= \frac{1}{2\pi\sigma^2} e^{\frac{-1}{2\sigma^2}\|\boldsymbol{r} - a_1\boldsymbol{h}_1 - a_2\boldsymbol{h}_2\|^2} \\ &= \frac{1}{2\pi\sigma^2} e^{\frac{-1}{2\sigma^2}(\|\boldsymbol{r}\|^2 + \|\boldsymbol{h}_1\|^2 + \|\boldsymbol{h}_2\|^2)} e^{\frac{1}{\sigma^2}(a_1 y_1 + a_2 y_2 - a_1 a_2 \rho)} \\ &= K e^{\frac{-1}{2\sigma^2}(-2a_1 y_1 - 2a_2 y_2 + 2a_1 a_2 \rho)}, \end{aligned} \tag{S.583}$$

where K is a constant, independent of a_1 and a_2. Plugging this into (S.582) yields:

$$\begin{aligned} \hat{a}_1 &= \text{sign}\left\{\log\frac{h\left\{Ke^{\frac{1}{\sigma^2}(y_1 + y_2 - \rho)}, Ke^{\frac{1}{\sigma^2}(y_1 - y_2 + \rho)}\right\}}{h\left\{Ke^{\frac{1}{\sigma^2}(-y_1 + y_2 + \rho)}, Ke^{\frac{1}{\sigma^2}(-y_1 - y_2 - \rho)}\right\}}\right\} \\ &= \text{sign}\left\{\log\frac{Ke^{\frac{y_1}{\sigma^2}} h\left\{e^{\frac{1}{\sigma^2}(y_2 - \rho)}, e^{\frac{-1}{\sigma^2}(y_2 - \rho)}\right\}}{Ke^{\frac{-y_1}{\sigma^2}} h\left\{e^{\frac{1}{\sigma^2}(y_2 + \rho)}, e^{\frac{-1}{\sigma^2}(y_2 + \rho)}\right\}}\right\}, \end{aligned} \tag{S.584}$$

where we used the fact that $h(Ax, Ay) = Ah(x, y)$ for both definitions of $h(\cdot\,,\cdot)$ and for any constaint A. The above equation simplifies to:

$$\hat{a}_1 = \text{sign}\left\{ y_1 + \left(\frac{\sigma^2}{2}\right)\log\frac{h\left\{e^{\frac{1}{\sigma^2}(y_2-\rho)}, e^{\frac{-1}{\sigma^2}(y_2-\rho)}\right\}}{h\left\{e^{\frac{1}{\sigma^2}(y_2+\rho)}, e^{\frac{-1}{\sigma^2}(y_2+\rho)}\right\}}\right\}. \tag{S.585}$$

(a) For the special case of JML, where $h_{JML}(x, y) = \max(x, y)$, this reduces to:

$$\begin{aligned}
\hat{a}_1^{JML} &= \text{sign}\left\{ y_1 + \left(\frac{\sigma^2}{2}\right)\log\frac{\max\left\{e^{\frac{1}{\sigma^2}(y_2-\rho)}, e^{\frac{-1}{\sigma^2}(y_2-\rho)}\right\}}{\max\left\{e^{\frac{1}{\sigma^2}(y_2+\rho)}, e^{\frac{-1}{\sigma^2}(y_2+\rho)}\right\}}\right\} \\
&= \text{sign}\left\{ y_1 + \left(\frac{\sigma^2}{2}\right)\log\frac{e^{\left|\frac{y_2-\rho}{\sigma^2}\right|}}{e^{\left|\frac{y_2+\rho}{\sigma^2}\right|}}\right\} \\
&= \text{sign}\left\{ y_1 + \tfrac{1}{2}\,|y_2-\rho| - \tfrac{1}{2}\,|y_2+\rho| \right\}, \\
&= \text{sign}\left\{ y_1 + \tfrac{1}{2}g_0(y_2-\rho) - \tfrac{1}{2}g_0(y_2+\rho) \right\},
\end{aligned} \tag{S.586}$$

where we have introduced the nonlinearity $g_0(x) = |x|$.

For the special case of IML, where $h^{IML}(x, y) = x + y$, (S.585) reduces to:

$$\begin{aligned}
\hat{a}_1^{IML} &= \text{sign}\left\{ y_1 + \left(\frac{\sigma^2}{2}\right)\log\frac{e^{\frac{1}{\sigma^2}(y_2-\rho)} + e^{\frac{-1}{\sigma^2}(y_2-\rho)}}{e^{\frac{1}{\sigma^2}(y_2+\rho)} + e^{\frac{-1}{\sigma^2}(y_2+\rho)}}\right\} \\
&= \text{sign}\left\{ y_1 + \left(\frac{\sigma^2}{2}\right)\log\frac{2\cosh\left(\frac{y_2-\rho}{\sigma^2}\right)}{2\cosh\left(\frac{y_2+\rho}{\sigma^2}\right)}\right\} \\
&= \text{sign}\left\{ y_1 + \frac{\sigma^2}{2}\operatorname{logcosh}\left(\frac{y_2-\rho}{\sigma^2}\right) - \frac{\sigma^2}{2}\operatorname{logcosh}\left(\frac{y_2+\rho}{\sigma^2}\right) \right\}, \\
&= \text{sign}\left\{ y_1 + \tfrac{1}{2}g_\sigma(y_2-\rho) - \tfrac{1}{2}\,g_\sigma(y_2+\rho) \right\},
\end{aligned} \tag{S.587}$$

where $g_\sigma(x) = \sigma^2 \operatorname{logcosh}(x/\sigma^2)$.

(b) The function $g_\sigma(x)$ is defined by:

$$g_\sigma(x) = \sigma^2 \log\left(\frac{e^{x/\sigma^2} + e^{-x/\sigma^2}}{2}\right)$$
$$= \sigma^2 \log\left(\frac{e^{|x|/\sigma^2} + e^{-|x|/\sigma^2}}{2}\right). \tag{S.588}$$

But $e^{-|x|/\sigma^2}$ goes to zero as $\sigma \to 0$, so that:

$$g_\sigma(x) = \sigma^2 \log(e^{x/\sigma^2} + e^{-x/\sigma^2}) - \sigma^2 \log(2)$$
$$\to \sigma^2 \log(e^{|x|/\sigma^2}) = \log(e^{|x|/\sigma^2})^{\sigma^2}$$
$$= \log(e^{|x|}) = |x|. \tag{S.589}$$

It follows that $\hat{a}_1^{\text{IML}}$ and $\hat{a}_1^{\text{JML}}$ usually agree at high SNR.

(c) When $y_2 = \rho = \sigma = 1$, the JML and IML detectors reduce to:

$$\hat{a}_1^{\text{JML}} = \text{sign}\left\{ y_1 - 1 \right\}, \tag{S.590}$$

$$\hat{a}_1^{\text{IML}} = \text{sign}\left\{ y_1 + \frac{1}{2} g_1(0) - \frac{1}{2} g_1(2) \right\}$$
$$= \text{sign}\left\{ y_1 - 0.6625 \right\}. \tag{S.591}$$

So any y_1 satisfying $\frac{1}{2} g_1(2) \approx 0.6625 < y_1 < 1$ will result in $\hat{a}_1^{\text{IML}} = 1$ but $\hat{a}_1^{\text{JML}} = -1$.

Problem 10-10. Let $\mathbf{P}$ be the identity matrix with the first and i-th columns swapped, so that

$$\mathbf{H} = \left[\boldsymbol{h}_1\, \boldsymbol{h}_2 \ldots \boldsymbol{h}_i \ldots \boldsymbol{h}_N \right], \qquad \mathbf{HP} = \left[\boldsymbol{h}_i\, \boldsymbol{h}_2 \ldots \boldsymbol{h}_1 \ldots \boldsymbol{h}_N \right]. \tag{S.592}$$

Then applying the Gram-Schmidt orthonormalization procedure to the columns of $\mathbf{HP}$, in reverse order, yields:

$$\mathbf{HP} = \mathbf{QG}, \tag{S.593}$$

where the columns of $\mathbf{Q}$ are an orthonormal basis for the span of $\{\boldsymbol{h}_j\}$, and where $\mathbf{G}$ is a lower triangular matrix of expansion coefficients. In particular, at the very last step of the GS procedure, the first column of $\mathbf{Q}$ is found by normalizing the projection error when the first column of $\mathbf{HP}$ is projected onto the subspace spanned by the other columns:

$$\boldsymbol{q}_1 = \frac{\boldsymbol{h}_i - \boldsymbol{h}_i}{\|\boldsymbol{h}_i - \hat{\boldsymbol{h}}_i\|}. \tag{S.594}$$

Solving this equation for $\boldsymbol{h}_i$ gives:

$$\boldsymbol{h}_i = \|\boldsymbol{h}_i - \hat{\boldsymbol{h}}_i\| \boldsymbol{q}_1 + \hat{\boldsymbol{h}}_i. \tag{S.595}$$

This implies that $G_{1,1} = \| \boldsymbol{h}_i - \hat{\boldsymbol{h}}_i \|$. (The matrix $\mathbf{G}$ contains expansion coefficients, so that $G_{1,1}$ is the coefficient of $\boldsymbol{q}_1$ when $\boldsymbol{h}_i$ represented as a linear combination of the columns of $\mathbf{Q}$.) Now

$$\begin{aligned}\mathbf{R}^{-1} &= (\mathbf{H}^*\mathbf{H})^{-1} \\ &= ((\mathbf{QGP}^*)^*(\mathbf{QGP}^*))^{-1} \\ &= (\mathbf{PG}^*\mathbf{Q}^*\mathbf{QGP}^*)^{-1} \\ &= (\mathbf{PG}^{-1})(\mathbf{PG}^{-1})^* .\end{aligned} \tag{S.596}$$

We are interested in the $(i, i)^{\text{th}}$ entry of this matrix. Since this matrix is in the form $\mathbf{FF}^*$, the $(i, i)^{\text{th}}$ entry will be the squared norm of the cc-th row of $\mathbf{F}$, namely:

$$(\mathbf{R}^{-1})_{i,i} = \| \mathbf{f}_i \|^2 \tag{S.597}$$

where $\mathbf{f}_i$ denotes the i-th row of $\mathbf{F} = \mathbf{PG}^{-1}$. But this is just the first row of $\mathbf{G}^{-1}$, a lower triangular matrix (because $\mathbf{G}$ is lower triangular), so its first row will be:

$$\mathbf{f}_i = [1 / G_{1,1}, 0, 0 \dots 0]. \tag{S.598}$$

(The nonzero entry ensures that $\mathbf{GG}^{-1} = \mathbf{I}$.) Thus,

$$(\mathbf{R}^{-1})_{i,i} = \| \mathbf{f}_i \|^2 = \frac{1}{G_{1,1}^2} = \| \boldsymbol{h}_i - \hat{\boldsymbol{h}}_i \|^{-2} , \tag{S.599}$$

the desired result.

Problem 10-11. We start with the trivial equality:

$$\mathbf{H}^*\mathbf{HH}^* + N_0\mathbf{H}^* = \mathbf{H}^*\mathbf{HH}^* + N_0\mathbf{H}^* \tag{S.600}$$

The fact that matrix multiplication is associative allows us to put parenthesis in different places on the left-hand side and right-hand side, yielding:

$$\mathbf{H}^*(\mathbf{HH}^* + N_0\mathbf{I}) = (\mathbf{H}^*\mathbf{H} + N_0\mathbf{I})\mathbf{H}^* \tag{S.601}$$

Right-multiplying both sides of the equation by $(\mathbf{HH}^* + N_0\mathbf{I})^{-1}$, and left-multiplying both sides by $(\mathbf{H}^*\mathbf{H} + N_0\mathbf{I})^{-1}$, yields:

$$(\mathbf{H}^*\mathbf{H} + N_0\mathbf{I})^{-1}\mathbf{H}^* = \mathbf{H}^*(\mathbf{HH}^* + N_0\mathbf{I})^{-1} , \tag{S.602}$$

the desired result.

Problem 10-12. We begin with the channel model $\boldsymbol{r} = \mathbf{H}\boldsymbol{a} + \boldsymbol{w}$, where $\mathrm{E}[\boldsymbol{w}\boldsymbol{w}^*] = \sigma^2\mathbf{I}$. Let $\boldsymbol{c}_i$ denote the i-th row of $\mathbf{C} = (\mathbf{H}^*\mathbf{H})^{-1}\mathbf{H}^*$, the pseudoinverse of $\mathbf{H}$. Then:

$$\boldsymbol{y} = \mathbf{C}\boldsymbol{r} = \boldsymbol{a} + \boldsymbol{n}, \tag{S.603}$$

where $\boldsymbol{n} = \mathbf{C}\boldsymbol{w}$ is correlated noise, and where the i-th noise component is $n_i = \boldsymbol{c}_i\boldsymbol{w}$.

(a) the optimal choice for i_1 minimizes the variance of n_{i_1}:

$$\begin{aligned} \mathrm{E}[\,|\,n_{i_1}|^2] &= \mathrm{tr}\{\mathrm{E}[\boldsymbol{c}_{i_1}\boldsymbol{w}\boldsymbol{w}^*\boldsymbol{c}_{i_1}^*]\} \\ &= \mathrm{tr}\{\boldsymbol{c}_{i_1}\mathrm{E}[\boldsymbol{w}\boldsymbol{w}^*]\boldsymbol{c}_{i_1}^*\} \\ &= \mathrm{tr}\{\boldsymbol{c}_{i_1}\sigma^2\mathbf{I}\boldsymbol{c}_{i_1}^*\} \\ &= \sigma^2\mathrm{tr}\{\boldsymbol{c}_{i_1}\boldsymbol{c}_{i_1}^*\} \\ &= \sigma^2\|\,\boldsymbol{c}_{i_1}\|^2 . \end{aligned} \tag{S.604}$$

So the best choice for i_1 is the index of the pseudoinverse row with minimum norm.

(b) The best choice for i_2 is the one that minimizes the variance of the noise at the input to the second slicer, assuming that the first decision was correct. From Fig. 10-31, the predicted value of the second noise component is:

$$\begin{aligned} \hat{n}_{i_2} &= p_{2,1}n_{i_1} . \\ &= p_{2,1}\boldsymbol{c}_{i_1}\boldsymbol{w}. \end{aligned} \tag{S.605}$$

The variance of the prediction error is the variance of the noise at the input to the second slicer, namely:

$$\begin{aligned} \mathrm{E}[\,|\,n_{i_2} - \hat{n}_{i_2}|^2\,] &= \mathrm{E}[\,|\,\boldsymbol{c}_{i_2}\boldsymbol{w} - p_{2,1}\boldsymbol{c}_{i_1}\boldsymbol{w}\,|^2] \\ &= \mathrm{E}[\,|\,(\boldsymbol{c}_{i_2} - p_{2,1}\boldsymbol{c}_{i_1})\boldsymbol{w}\,|^2] \\ &= \sigma^2\|\,\boldsymbol{c}_{i_2} - p_{2,1}\boldsymbol{c}_{i_1}\|^2 . \end{aligned} \tag{S.606}$$

An optimal choice for the prediction coefficient will force $p_{2,1}\boldsymbol{c}_{i_1}$ to be as close to $\boldsymbol{c}_{i_2}$ as possible, in a Euclidean distance sense. Therefore, the best choice for i_2 satisfies (10.160).

(c) The best choice for i_k minimizes the variance at the input to the k-th slicer, assuming previous decisions are correct. Specifically, the best choice for i_k minimizes:

$$\mathrm{E}[\,|\,n_{i_k} - \hat{n}_{i_k}|^2\,], \tag{S.607}$$

where

$$\hat{n}_{i_k} = \sum_{j<k} p_{k,j}\, n_{i_j} . \tag{S.608}$$

Substituting (S.608) into (S.607) yields:

$$
\begin{aligned}
\mathrm{E}[\,|\,n_{i_k} - \hat{n}_{i_k}|^2\,] &= \mathrm{E}[\,|\,n_{i_k} - \textstyle\sum_{j<k} p_{k,j}\, n_{i_j}\,|^2] \\
&= \mathrm{E}[\,|\,(\boldsymbol{c}_{i_k} - \textstyle\sum_{j<k} p_{k,j}\, \boldsymbol{c}_{i_j})\boldsymbol{w}\,|^2] \\
&= \sigma^2 \|\,\boldsymbol{c}_{i_k} - \textstyle\sum_{j<k} p_{k,j}\, \boldsymbol{c}_{i_j}\|^2 \\
&= \sigma^2 \|\,\boldsymbol{c}_{i_k} - \hat{\boldsymbol{c}}_{i_k}\|^2
\end{aligned} \tag{S.609}
$$

where we have introduced

$$
\hat{\boldsymbol{c}}_{i_k} = \sum_{j<k} p_{k,j}\, \boldsymbol{c}_{i_j}\,. \tag{S.610}
$$

For a given i_k, the prediction coefficients $\{p_{k,j}\}$ are chosen to minimize (S.609). In other words, they are chosen so that $\hat{\boldsymbol{c}}_{i_k}$ is as close to $\boldsymbol{c}_{i_k}$ as possible, which is achieved when $\hat{\boldsymbol{c}}_{i_k}$ is the *projection* of $\boldsymbol{c}_{i_k}$ onto the linear subspace spanned by $\{\boldsymbol{c}_{i_1}, \ldots\, \boldsymbol{c}_{i_{k-1}}\}$. Therefore, (10.161) follows.

Problem 10-13. The QR decomposition of the matrix **H** with its columns in the natural order yields a **G** matrix whose first diagonal element is G_{11}, which is positive and real. If y_1 is the first output of the whitened matched filter (10.129) , then it may be expressed as: $y_1 = G_{11}a_1 + n_1$, where n_1 is a Gaussian random variable with variance $\sigma^2 = 1$. The probability that the best-first sphere detector extends the node that does not belong to the transmitted path can be expressed as:

$$P_{e1} = \Pr[\,|\,y_1 - G_{11}a_1\,|^2 > |\,y_1 + G_{11}a_1\,|^2\,]. \tag{S.611}$$

After substituting for y_1, this probability becomes:

$$\begin{aligned} \Pr[\,|\,n_1\,|^2 > |\,n_1 + 2G_{11}a_1\,|^2\,] &= \Pr[\,G_{11} + n_1a_1 < 0] \\ &= \Pr[\,n_1 > G_{11}] \\ &= Q(G_{11}). \end{aligned} \tag{S.612}$$

(a) From Example 10-24, $G_{11} = 0.77$ with the natural ordering. Therefore, $P_{e1}^{(a)} = 0.22$.

(b) From Example 10-24, $G_{11} = 1.22$ with the BLAST ordering. Therefore, $P_{e1}^{(b)} = 0.11$, about half of that from part (a).

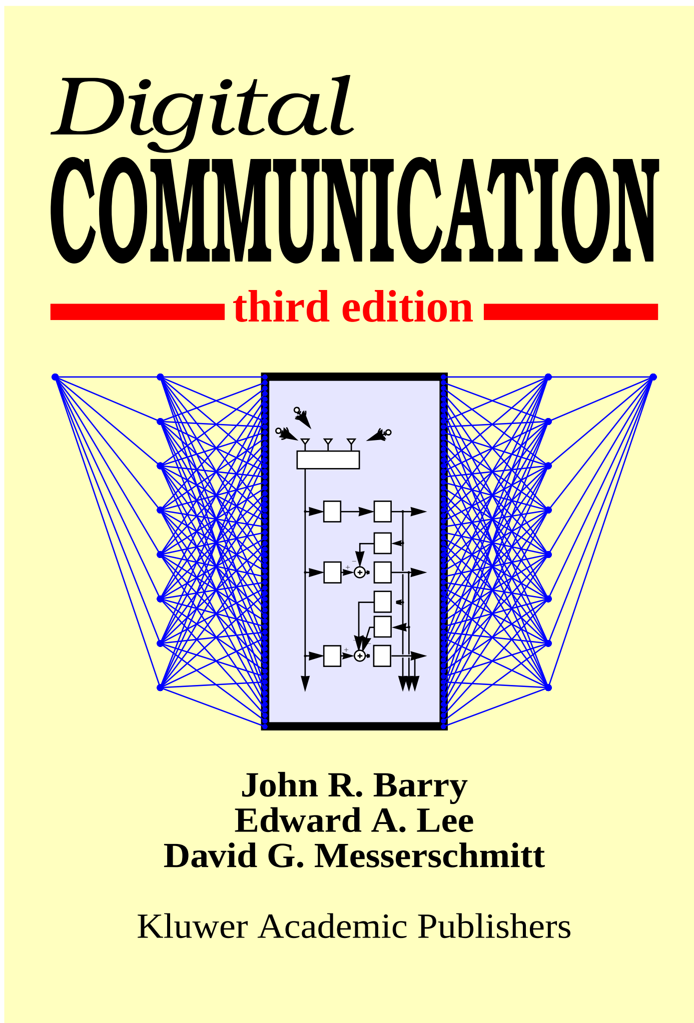
(c) First of all let's assume that the decision reached by the best-first sphere detector is the correct decision vector. The complexity of the sphere detector is directly related to how many nodes are visited by the search algorithm, and indirectly related to how quickly the cost threshold J is reduced.

If the best-first sphere detector extends to the correct node at the first stage, then it will *search the correct side of the tree first*, thereby establishing a low cost threshold J for its tentative decision. With this small cost threshold, searching the wrong side of the tree becomes less complex since many of its nodes will have costs greater than J allowing them and their subbranches to be pruned.

From part (a), the best-first sphere detector is half as likely to extend the wrong node at the first stage when the BLAST ordering is used instead of the natural ordering. Therefore, with the BLAST ordering, the best-first sphere detector is half as likely to waste unnecessary computations searching the wrong side of the tree.

Solutions Manual

for



Chapter 11

Problem 11-1. Starting with the MISO model of (11.36) and (11.37):

$$r = \mathbf{H}\boldsymbol{a} + n = \mathbf{H}\boldsymbol{w}x + n\ , \tag{S.613}$$

the instantaneous SNR per receive antenna is

$$\begin{aligned} SNR &= \frac{E[|\mathbf{H}\boldsymbol{w}|^2|x|^2]}{E[|n|^2]} \\ &= |\,\mathbf{H}\boldsymbol{w}\,|^2 E/N_0\ , \end{aligned} \tag{S.614}$$

where $E = \mathrm{E}[\,|\,x\,|^2]$ is the average symbol energy. Application of the Cauchy-Schwartz inequality yields:

$$\text{SNR} \le \|\,\mathbf{H}\,\|^2\|\,\boldsymbol{w}\,\|^2 E/N_0 = \|\,\mathbf{H}\,\|^2 E/N_0\,, \tag{S.615}$$

where the equality follows from the constraint that $\|\,\boldsymbol{w}\,\| = 1$. Thus, regardless of how we choose $\boldsymbol{w}$, the SNR cannot exceed the right-hand side of (S.615). But we can achieve this maximum by setting $\boldsymbol{w} = \mathbf{H}^*/\|\,\mathbf{H}\,\|$, which follows from the Cauchy-Schwartz inequality, as verified below:

$$SNR = |\,\mathbf{H}\boldsymbol{w}\,|^2 E/N_0 = \left|\,\mathbf{H}\frac{\mathbf{H}^*}{\|\mathbf{H}\|}\,\right|^2 E/N_0 = \|\,\mathbf{H}\,\|^2 E/N_0\ . \tag{S.616}$$

Note that the solution is not unique; setting $\boldsymbol{w} = e^{j\phi}\mathbf{H}^*/\|\,\mathbf{H}\,\|$ for any angle ϕ will also maximize SNR.

Problem 11-2. Observe that both $\boldsymbol{x} = \frac{1}{2}[1,-1,\ 1,\ -1]^T$ and $\boldsymbol{y} = \frac{1}{2}[1,1,\ -1,\ -1]^T$ are unit-length, satisfying $\|\,\boldsymbol{x}\,\| = \|\,\boldsymbol{y}\,\| = 1$. Furthermore, they are orthogonal, since $\boldsymbol{y}^*\boldsymbol{x} = \sum_i x_i y_i^* = 0$. Therefore, $\boldsymbol{x}$ and $\boldsymbol{y}$ form an orthonormal basis for their span. The projection $\hat{\boldsymbol{n}}$ of $\boldsymbol{n}$ onto this subspace is then:

$$\begin{aligned} \hat{\boldsymbol{n}} &= \boldsymbol{x}(\boldsymbol{x}^*\boldsymbol{n}) + \boldsymbol{y}(\boldsymbol{y}^*\boldsymbol{n}) \\ &= \mathbf{Q}\mathbf{Q}^*\boldsymbol{n}\ , \end{aligned} \tag{S.617}$$

where $\mathbf{Q} = [\ \boldsymbol{x},\ \boldsymbol{y}\ ]$. Hitting both sides on the left by $\mathbf{Q}^*$ yields:

$$\mathbf{Q}^*\hat{\boldsymbol{n}} = \mathbf{Q}^*\boldsymbol{n}\ . \tag{S.618}$$

Since the components of $\boldsymbol{n}$ are i.i.d. $\mathcal{CN}(0,\ N_0)$, the components of $\boldsymbol{u} = [u_1,\ u_2]^T = \mathbf{Q}^*\boldsymbol{n}$ are zero mean, jointly Gaussian, and correlated according to:

$$\begin{aligned} \mathbf{R}_{\boldsymbol{uu}} = \mathrm{E}[\boldsymbol{uu}^*] &= \mathbf{Q}^*\mathrm{E}[\boldsymbol{nn}^*]\mathbf{Q} \\ = \mathbf{Q}^*N_0\mathbf{I}\mathbf{Q} &= N_0\mathbf{Q}^*\mathbf{Q} \end{aligned}$$

$$= N_0 \begin{bmatrix} \boldsymbol{x}^* \\ \boldsymbol{y}^* \end{bmatrix} \begin{bmatrix} \boldsymbol{x} & \boldsymbol{y} \end{bmatrix} = N_0 \mathbf{I} \,. \tag{S.619}$$

This implies that u_1 and u_2 are i.i.d. $\mathcal{CN}(0, N_0)$, just like the components of $\boldsymbol{n}$. It follows that:

$$\begin{aligned} \| \hat{\boldsymbol{n}} \|^2 &= \| u_1 \boldsymbol{x} + u_2 \boldsymbol{y} \|^2 \\ &= \| u_1 \boldsymbol{x} \|^2 + \| u_2 \boldsymbol{y} \|^2 + 2\mathrm{Re}\{ u_1 u_2{}^* \boldsymbol{y}^* \boldsymbol{x} \} \\ &= | u_2 |^2 + | u_2 |^2 = \| \boldsymbol{u} \|^2. \end{aligned} \tag{S.620}$$

Since u_1 and u_2 are i.i.d. $\mathcal{CN}(0, N_0)$, it follows that $\| \hat{\boldsymbol{n}} \|^2$ *is a central chi-squared random variable with mean* $2N_0$ *and with four degrees of freedom.* Its pdf is given by:

$$f_{\|\hat{n}\|^2}(x) = \frac{x}{N_0^2} e^{-x/N_0} u(x) \,. \tag{S.621}$$

Problem 11-3. Equation (11.24) gives the following expression for the BER:

$$BER = \frac{1}{2} \sum_{k=0}^{m} (-1)^k \binom{m}{k} \left(1 + \frac{k}{E_b/N_0} \right)^{-1/2} . \tag{S.622}$$

Using the expansion for $(1+x)^{-1/2}$, with $x = \frac{k}{E_b/N_0}$, yields:

$$(1+x)^{-1/2} = \sum_{l=0}^{\infty} \frac{(2l-1)!!}{(2l)\ \ !!} (-x)^l = \sum_{l=0}^{\infty} \frac{(2l-1)!!}{(2l)\ \ !!} \left(-\frac{k}{E_b/N_0} \right)^l . \tag{S.623}$$

Substituting into (11.24), we get

$$\begin{aligned} BER &= \frac{1}{2} \sum_{k=0}^{m} (-1)^k \binom{m}{k} \sum_{l=0}^{\infty} \frac{(2l-1)!!}{(2l)\ \ !!} \left(-\frac{k}{E_b/N_0} \right)^l \\ &= \frac{1}{2} \sum_{l=0}^{\infty} \frac{(2l-1)!!}{(2l)\ \ !!} (-E_b/N_0)^{-l} \sum_{k=0}^{m} (-1)^k \binom{m}{k} (k)^l . \end{aligned} \tag{S.624}$$

At high SNR, the infinite sum in the above expression can be approximated by truncating to the first *m* terms, since $(E_b/N_0)^{-l} \to 0$ as $l \to \infty$:

$$BER \approx \frac{1}{2} \sum_{l=0}^{m} \frac{(2l-1)!!}{(2l)\ \ !!} (-E_b/N_0)^{-l} \sum_{k=0}^{m} (-1)^k \binom{m}{k} (k)^l . \tag{S.625}$$

But the second summation can be written as follows

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} (k)^l = \begin{cases} 0 & \mathit{if} \quad l \in \{0, 1, \ldots, m-1\} \\ (-1)^m m! & \mathit{if} \quad l = m \end{cases} . \tag{S.626}$$

Hence, the BER simplifies:

$$BER \approx \frac{1}{2}\frac{(2m-1)!!}{(2m)\quad !!}(E_b/N_0)^{-m}m! \tag{S.627}$$

or equivalently:

$$BER \approx \frac{A_m}{(E_b/N_0)^m}, \tag{S.628}$$

where

$$A_m = \frac{1}{2}\frac{(2m-1)!!}{(2m)\quad !!}m! = \frac{1}{2}\frac{(2m-1)!!}{2^m}. \tag{S.629}$$

Equivalently,

$$A_m = \left(m - \frac{1}{2}\right)A_{m-1} \text{ with } A_1 = \left(\frac{1}{4}\right). \tag{S.630}$$

Problem 11-4.

(a) The MRC receiver processing is given by

$$y = \mathbf{H}^*\boldsymbol{r} = \mathbf{H}^*(\mathbf{H}a + \boldsymbol{n}). \tag{S.631}$$

The instantaneous SNR per receive antenna per bit is:

$$SNR_{\text{bit}} = \frac{\|H^*H\|^2 E_b}{\|H^*\|^2 N_0} = \frac{\|H\|^2 E_b}{N_0} = \frac{E_b}{N_0}\sum_{i=1}^{m}|h_i|^2. \tag{S.632}$$

The average post-detection SNR is:

$$\mathrm{E}[SNR_{\text{bit}}] = \mathrm{E}\left[\frac{E_b}{N_0}\sum_{i=1}^{m}|h_i|^2\right] = \mathrm{E}\left[\frac{E_b}{N_0}\sum_{i=1}^{m}|h_i|^2\right] = \frac{E_b}{N_0}\sum_{i=1}^{m}(1) = \frac{mE_b}{N_0}. \tag{S.633}$$

Hence, the average post-detection SNR is increased *m*-fold using MRC combining.

(b) For selection combining, the instantaneous SNR per receive antenna per bit is given by

$$SNR_{\text{bit}} = \frac{E_b}{N_0}\max_i\{|h_i|^2\} \tag{S.634}$$

The average post-detection SNR is:

$$\mathrm{E}[SNR_{\text{bit}}] = \frac{E_b}{N_0}E\left[\max_i\{|h_i|^2\}\right]. \tag{S.635}$$

The probability density function of $X = \max_i\{|h_i|^2\}$ is given by (11.22):

$$f_X(x) = \sum_{k=1}^{m} (-1)^{k-1} \binom{m}{k} k e^{-kx} u(x) . \tag{S.636}$$

Thus, the average post detection SNR can be computed by integrating:

$$\begin{aligned} \mathrm{E}[SNR_{\mathrm{bit}}] &= \frac{E_b}{N_0} \int_0^{\infty} x \cdot f_X(x) dx \\ &= \frac{E_b}{N_0} \int_0^{\infty} x \sum_{k=1}^{m} (-1)^{k-1} \binom{m}{k} k e^{-kx} dx \end{aligned} \tag{S.637}$$

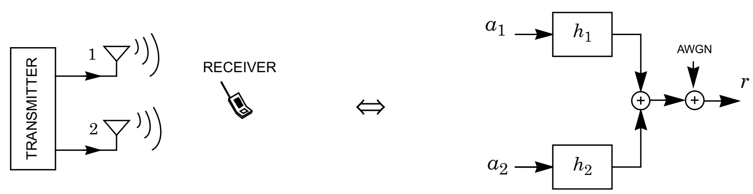
$$= \frac{E_b}{N_0} \sum_{k=1}^{m} (-1)^{k-1} \binom{m}{k} \left\{ k \int_0^{\infty} x e^{-kx} dx \right\} . \tag{S.638}$$

But the integral inside the brackets is the mean value of an exponential random variable with mean $1/k$. (Or we can just do the integration directly, it's simple.) Thus, we have:

$$\begin{aligned} \mathrm{E}[SNR_{\mathrm{bit}}] &= \frac{E_b}{N_0} \sum_{k=1}^{m} (-1)^{k-1} \binom{m}{k} \left\{ \frac{1}{k} \right\} \\ &= \frac{E_b}{N_0} \sum_{k=1}^{m} \frac{1}{k} . \end{aligned} \tag{S.639}$$

Hence, selection combining increases the average post-detection SNR by a factor of $\sum_{k=1}^{m} \frac{1}{k}$.

Problem 11-5. The starting point is the narrowband two-input one-output channel:



The received sample is:

$$r = h_1 a_1 + h_2 a_2 + n \ , \tag{S.640}$$

where $n \sim \mathcal{CN}(0, N_0)$. The premise of this problem is that the transmitter knows the gains h_1 and h_2, and it transmits a single 4-QAM symbol $a \in \{\pm 1 \pm j\}\sqrt{E_b}$ across the antenna with the higher gain, and transmits nothing across the antenna with the lower gain. With this transmitter strategy, the received sample reduces to:

$$r = h_{i_{\max}} a + n \ , \tag{S.641}$$

where $i_{\max} = \arg\max_i\{\,|h_i|\,\}$. This is exactly the same as the simplified channel model after selection combining at the receiver; see (11.18). *Antenna selection at the transmitter leads to exactly the same channel model as antenna selection at the receiver*. So we need not analyze this anew, we can reuse the results from section 11.3.2 directly. In particular, the average bit-error rate at high SNR is given by (11.25) with $m = 2$, namely:

$$BER \approx \frac{3/8}{(E_b/N_0)^2} \,. \tag{S.642}$$

Problem 11-6. No changes would be needed. From the perspective of the receiver, a second antenna — transmitting a delayed version of the signal transmitted by the first antenna — would appear as an extra multipath component. And an OFDM receiver is already well-equipped to mitigate multipath-induced ISI. The only constraint is that the delay between antenna one and antenna two be small enough so that, when combined with the delay spread of the channel, it does not exceed the cyclic prefix length.

Problem 11-7. All of the space-time codewords have two rows, which implies that there are two transmit antennas, and they all have two columns, which implies that the duration of each codeword is two signaling intervals. If m denotes the number of receive antennas, then a space-time code achieves "full" diversity when the maximum diversity order $2m$ is achieved.

(a) To show that a given space-time code does not achieve full diversity, it is sufficient to show that it violates the rank criterion of section 11.5.4; i.e., that there exists two valid input vectors $\boldsymbol{x}$ and $\boldsymbol{x}'$ for which the difference matrix $\mathbf{A}(\boldsymbol{x}) - \mathbf{A}(\boldsymbol{x}')$ is not full rank.

- $\mathbf{A}_1$ is not full diversity.
  It fails the rank criterion test when $\boldsymbol{x} = [1 + j,\ 1 + j]^T$ and $\boldsymbol{x}' = [1 + j,\ 1 + j]^T$:

$$\mathbf{A}_1\left(\begin{bmatrix} 1+j \\ 1+j \end{bmatrix}\right) - \mathbf{A}_1\left(\begin{bmatrix} -1+j \\ -1+j \end{bmatrix}\right) = \begin{bmatrix} 1+j & 1-j \\ 1+j & 1+j \end{bmatrix} - \begin{bmatrix} -1+j & -1-j \\ -1+j & -1+j \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}, \tag{S.643}$$

  which has rank one.

- $\mathbf{A}_2$ is not full diversity.
  It fails the rank criterion test when $\boldsymbol{x} = [1 + j,\ 1 + j]^T$ and $\boldsymbol{x}' = [-1 + j,\ 1 - j]^T$:

$$\mathbf{A}_2\left(\begin{bmatrix} 1+j \\ 1+j \end{bmatrix}\right) - \mathbf{A}_2\left(\begin{bmatrix} -1+j \\ 1-j \end{bmatrix}\right) = \begin{bmatrix} 1+j & 1+j \\ 1+j & -1+j \end{bmatrix} - \begin{bmatrix} -1+j & 1-j \\ 1-j & 1+j \end{bmatrix} = \begin{bmatrix} 2 & 2j \\ 2j & -2 \end{bmatrix}, \tag{S.644}$$

  which has rank one.

- $\mathbf{A}_3$ is full diversity. Observe that $\mathbf{A}_3 = \mathbf{A}^*$, where $\mathbf{A}$ is the Alamouti space-time encoder of section 11.5.3. Since $\mathbf{A}$ satisfies the rank criterion, so too does $\mathbf{A}_3$. (The Hermitian transpose operator does not change the rank of a matrix.)

- $\mathbf{A}_4$ is not full diversity.
  It fails the rank criterion test when $\boldsymbol{x} = [1 + j,\ 1 - j]^T$ and $\boldsymbol{x}' = [-1 + j,\ -1 - j]^T$:

$$\mathbf{A}_4\left(\begin{bmatrix} 1+j \\ 1-j \end{bmatrix}\right) - \mathbf{A}_4\left(\begin{bmatrix} -1+j \\ -1-j \end{bmatrix}\right) = \begin{bmatrix} 1+j & 1-j \\ 1-j & 1+j \end{bmatrix} - \begin{bmatrix} -1+j & -1-j \\ -1-j & -1+j \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}, \tag{S.645}$$

  which has rank one.

(b) Only $\mathbf{A}_3$ achieves full diversity, so we need only find the coding gain for $\mathbf{A}_3$. The coding gain of a full-rank code is the smallest (for all $i \neq j$) geometric mean of the eigenvalues of the Hermitian matrix $(\mathbf{A}_i - \mathbf{A}_j)(\mathbf{A}_i - \mathbf{A}_j)^*$. In Example 11-5 it was demonstrated that the coding gain of the Alamouti code is 1. Since $\mathbf{A}_3$ is the conjugate transpose of the Alamouti code, and since the conjugate transpose does not change the eigenvalues of a Hermitian matrix, it follows that the coding gain of $\mathbf{A}_3$ is also 1.

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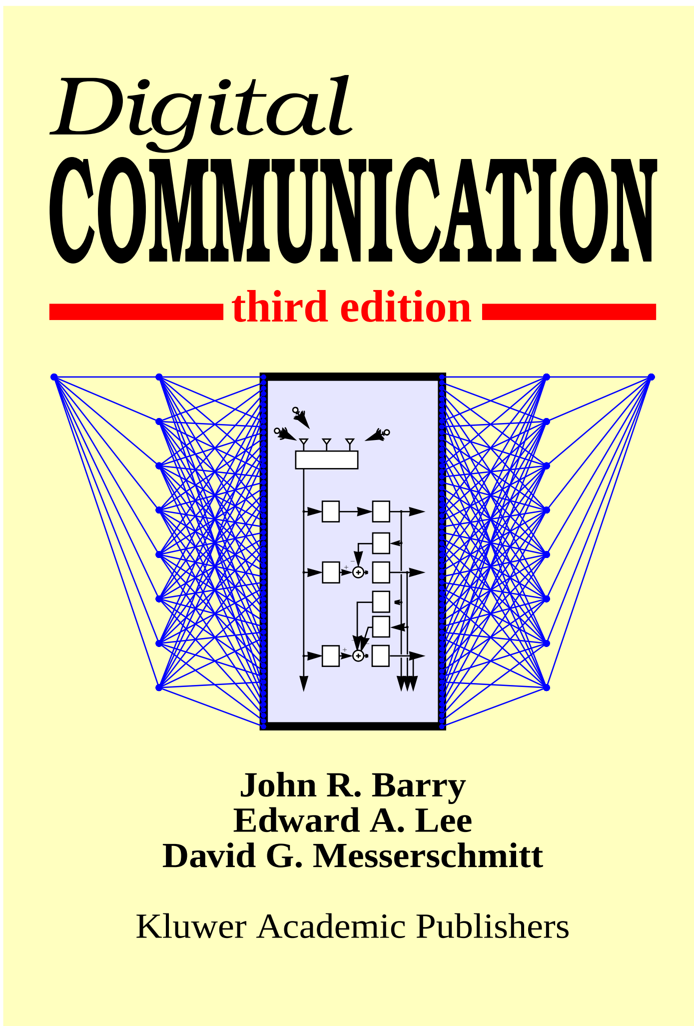
for



Chapter 12

Problem 12-1.

(a) There are only two codewords $00\ldots0$ and $11\ldots1$, which have distance n, so $d_{H,\min} = n$.

(b) The (7, 4) Hamming code has $d_{H,\min} = 3$, and its rate $4/7$ code. A minimum distance of $d_{H,\min} = 3$ in a repetition code requires $n = 3$, which has rate $1/3$, considerably worse than $4/7$.

Problem 12-2.

(a) For all $\boldsymbol{c} \in \mathcal{C}$, $\boldsymbol{c}\mathbf{H}^T = \mathbf{0}$. The product $\boldsymbol{c}\mathbf{H}^T$ is a linear combination of rows of $\mathbf{H}^T$, or columns of $\mathbf{H}$. Hence the minimum number of columns of $\mathbf{H}$ that can be added to produce $\mathbf{0}$ is

$$\min_{\substack{\boldsymbol{c} \in \mathcal{C} \\ \boldsymbol{c} \neq \mathbf{0}}} \boldsymbol{w}_H(\boldsymbol{c}) \tag{S.613}$$

which equals $d_{H,\min}$.

(b) From part (a), no linear combination of $d_{H,\min} - 1$ or fewer columns of $\mathbf{H}$ can be zero, so $\mathbf{H}$ has rank $d_{H,\min} - 1$. Since $\mathbf{H}$ has dimension $n \times (n - k)$, its rank cannot be larger than $n - k$, so

$$d_{H,\min} - 1 \leq n - k \,, \tag{S.614}$$

from which the result follows.

Problem 12-3.

(a) Reverse the order of the first three columns of $\mathbf{G}$.

(b) The resulting parity-check matrix is:

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{S.615}$$

(c) In each case the most likely error pattern is the one with the smallest Hamming weight:

$\boldsymbol{s}$	$\hat{\boldsymbol{e}}$	$\boldsymbol{s}$	$\hat{\boldsymbol{e}}$
0000	000 0000	1000	000 1000
0001	000 0001	1001	000 1001
0010	000 0010	1010	000 1010
0011	000 0011	1011	000 1011
0100	000 0100	1100	100 0001
0101	000 0101	1101	100 0000
0110	000 0110	1110	001 0000
0111	010 0000	1111	100 0010

For those entries with two or three one bits in $\hat{\boldsymbol{e}}$, the choices for $\hat{\boldsymbol{e}}$ may not be unique. Syndromes for which this is the case correspond to errors that cannot be reliably corrected, but can be detected.

(d) This code is equivalent to the dual of the $(7, 4)$ Hamming code.

(e) $d_{H,\min} = 4$, so only one bit error can be reliably corrected.

(f) $\boldsymbol{c} = 1010011$.

Problem 12-4. In this case, $m = n - k = 4$, and the parity-check matrix has as columns all possible 4-bit patterns except the all zero pattern. To get it into systematic form, simply arrange the columns so that the last four columns form an identity matrix. The first 11 columns can appear in any order.

(a) One example of a systematic parity-check matrix:

$$\mathbf{H} = \begin{bmatrix} 110100011111000 \\ 110111110000100 \\ 111001100110010 \\ 101110101010001 \end{bmatrix} \tag{S.616}$$

Since this is in the form $\mathbf{H} = [\mathbf{P}, \mathbf{I}]$, a valid generator matrix is $\mathbf{G} = [\mathbf{I}, -\mathbf{P}^T]$:

$$\mathbf{G} = \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0&1&1&1&1 \\ 0&1&0&0&0&0&0&0&0&0&0&1&1&1&0 \\ 0&0&1&0&0&0&0&0&0&0&0&0&0&1&1 \\ 0&0&0&1&0&0&0&0&0&0&0&1&1&0&1 \\ 0&0&0&0&1&0&0&0&0&0&0&0&1&0&1 \\ 0&0&0&0&0&1&0&0&0&0&0&0&1&1&0 \\ 0&0&0&0&0&0&1&0&0&0&0&0&1&1&1 \\ 0&0&0&0&0&0&0&1&0&0&0&1&1&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0&1&0&0&1 \\ 0&0&0&0&0&0&0&0&0&1&0&1&0&1&0 \\ 0&0&0&0&0&0&0&0&0&0&1&1&0&1&1 \end{bmatrix}. \tag{S.617}$$

(b) The parity-check matrix is

$$\mathbf{H} = \begin{bmatrix} 000000011111111 \\ 000111100001111 \\ 011001100110011 \\ 101010101010101 \end{bmatrix} . \tag{S.618}$$

If $\boldsymbol{e}$ is all zero except for a 1 in position i, then $\boldsymbol{e}\mathbf{H}'$ will be the transpose of the i-th column of **H**, which from (S.618) is a binary representation of i.

Problem 12-5. This problem was solved in Example 12-11. From (12.31), the asymptotic coding gain is $Rd_{\text{min}} = (11/15) \times 3 = 33/15 = 3.42$ dB. In reality, from Fig. 12-7, we see that the coding gain is only 2.6 dB at a bit-error probability of 10^{-5}.

Problem 12-6. The rate is $R = 4/15$, and the minimum Hamming distance is $d_{\text{min}} = 8$, so that the asymptotic coding gain with soft decoding is:

$$Rd_{\text{min}} = (4/15) \times 8 = 31/15 = 3.29 \text{ dB}. \tag{S.619}$$

Problem 12-7.

(a) The rate system is described by the following equations

$$C^{(1)}(D) = B^{(1)}(D) \oplus B^{(2)}(D)(D \oplus D^2) \tag{S.620}$$

$$C^{(2)}(D) = B^{(1)}(D)D \oplus B^{(2)}(D)(1 \oplus D^2) \tag{S.621}$$

$$C^{(3)}(D) = B^(2)(D)D \ . \tag{S.622}$$

These equations can be manipulated to get

$$0 = C^{(1)}(D)D^2 \oplus C^{(2)}(D)D \oplus C^{(3)}(D)(1 \oplus D^3) , \tag{S.623}$$

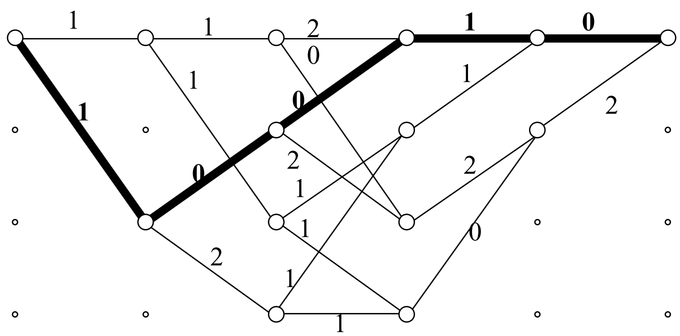
which means that the parity-check matrix is

$$\mathbf{H}(D) = [D^2,\ D,\ 1 \oplus D^3] \ . \tag{S.624}$$

(b) A similar technique can be used to derive the parity-check matrix, showing that it is the same.

Problem 12-8. The condition $m_k = 0$ for $k < 0$ implies that the starting state is the (0, 0) state. The condition $m_k = 0$ for $k \geq 3$ implies that the fifth state and beyond are zero. Hence

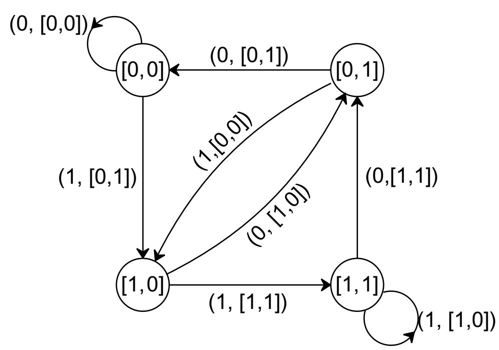
the trellis is shown in the following figure:

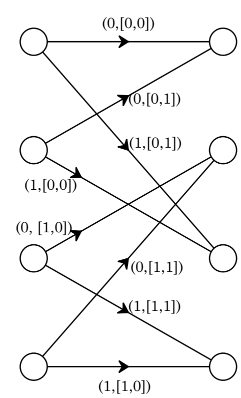


The transitions are labeled with their weights, which are the Hamming distances from the observation bit pairs. The minimum path metric corresponds to the bold path. Without errors introduced by the BSC, the observation would have been {1,1,0,1,1,1,0,0,0,0,...}, implying that two bit errors were made. The decision $\hat{m}_k$ is {1,0,0,...}.

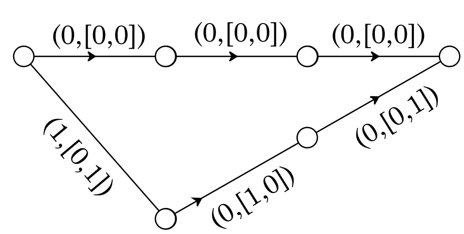
Problem 12-9.

(a) The state transition diagram and trellis are shown in the following figure:





The error event with the minimum Hamming distance is again an event with length $K = 2$, as shown in the following figure:

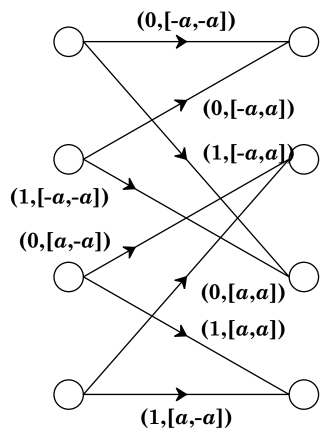


The minimum Hamming distance is therefore three. Assume a BSC with probability of correct transmission p. The probability of the error event is

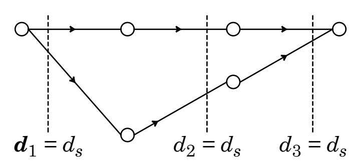
$$\binom{3}{2}p^2(1-p) + \binom{3}{1}p^3 = 3p^2(1-p) + p^3 \quad . \tag{S.625}$$

For p small this is approximately $3p^2$ which is significantly worse than the probability computed in (12.86), which for small p is approximately $10p^3$.

(b) The trellis is shown below re-labeled with the binary antipodal outputs.



The minimum distance error event is shown below.



The ML soft decoder therefore has $d_{min} = \sqrt{3}\, d_s$ where $d_s = 2a$. Hence, assuming the probability of error is dominated by this event,

$$\Pr[\text{error event}] \approx Q\left(\frac{\sqrt{3}a}{\sigma_c}\right), \tag{S.626}$$

where $\sigma_c{}^2$ is the variance of the noise of the coded system, which is twice the variance of the noise of the uncoded system, $\sigma_c{}^2 = 2\sigma_u{}^2$. Hence, the coded system is $10\log(3/2) \approx 1.8$ dB better than the uncoded system. This is a full 2.2 dB worse than the encoder of Fig. 12-10(a). This is not surprising because the minimum distance is far worse.

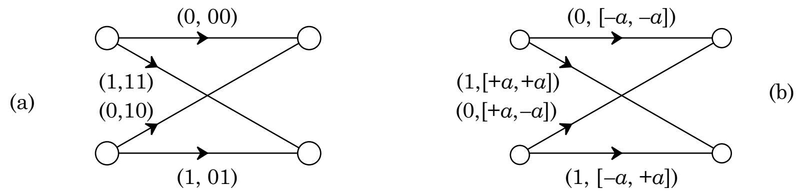


Fig. S-9. One stage of the trellis for the coder in Fig. 12-38. In (a) it is labeled with the binary output. In (b) it is labeled with the channel symbols, assuming binary antipodal signaling.

Problem 12-10. One stage of the trellis is shown in Fig. S-9.

(a) The minimum Hamming distance error event has distance 3, by inspection of Fig. S-9a. Following the dvelopment of (12.86) we get that the probability of this error event is bounded by

$$\Pr[\text{this error event}] \le \binom{3}{2} p^2(1-p) + \binom{3}{3} p^3 = 3p^2(1-p) + p^3 \ . \tag{S.627}$$

(b) By inspection of Fig. S-9b we see that the minimum Euclidean distance error event has distance $2a\sqrt{3}$. Hence

$$\Pr[\text{this error event}] \approx Q\left(\frac{2\sqrt{3}a}{2\sigma_c}\right) = Q\left(\frac{2\sqrt{3}a}{2\sqrt{2}\sigma_u}\right) . \tag{S.628}$$

The uncoded system has probability of error

$$\Pr[\text{error, uncoded}] = Q\left(\frac{a}{\sigma_c}\right) \tag{S.629}$$

so the coding gain is approximately

$$20\log\left(\frac{\sqrt{3}}{\sqrt{2}}\right) \approx 1.8 \text{ dB} \ . \tag{S.630}$$

Note that this is the same coding gain achieved by the coder in Fig. 12-37 with a soft decoder. This coder is simpler, however, since its trellis has only two states.

Problem 12-11. The code is not linear because it does not include the zero vector. (A linear code is by definition closed under addition, and adding any codeword to itself produces the zero vector.)

Problem 12-12. The length is $n = 7$, the dimension is $k = 3$, the minimum Hamming distance is $d_{H,\min} = 4$, and the codewords are:

```
0000000
1110100
0111010
1101001
1001110
0011101
1010011
0100111
```

Note that all of the non-zero codewords have the same weight.

Problem 12-13. Suppose that a message bit of $m = 0$ gets mapped to the the symbols $a_0 = a_1 = a_2 = -c$, while $m = 1$ gets mapped to $a_0 = a_1 = a_2 = +c$.

Consider first the soft (ML) decoder. It decides $\hat{m}_{\text{soft}} = 1$ if and only if $\|\boldsymbol{r} - \boldsymbol{s}\|^2 < \|\boldsymbol{r} + \boldsymbol{s}\|^2$, where $\boldsymbol{s} = [c, c, c]^T$. But:

$$\begin{aligned}\|\boldsymbol{r} - \boldsymbol{s}\|^2 - \|\boldsymbol{r} + \boldsymbol{s}\|^2 &= \|\boldsymbol{r}\|^2 + \|\boldsymbol{s}\|^2 - 2c\textstyle\sum_i r_i - (\|\boldsymbol{r}\|^2 + \|\boldsymbol{s}\|^2 + 2c\sum_i r_i) \\ &= -4c\textstyle\sum_i r_i\,. \end{aligned} \tag{S.631}$$

Therefore, the soft decoder decides $\hat{m}_{\text{soft}} = 1$ if and only if $\sum_i r_i > 0$.

On the other hand, the hard decoder decides $\hat{m}_{\text{hard}} = 0$ if and only if at least two of the three observations are negative. Therefore, we can achieve $\hat{m}_{\text{soft}} = 1$ and $\hat{m}_{\text{hard}} = 0$ by choosing:

$$r_0 = -0.1,\ r_1 = -0.1,\ r_2 = +0.3\,. \tag{S.632}$$

The sum is positive, even though the majority are negative.

Problem 12-14. The generator matrix has one row and $n = 2^{\mu+1} - 1$ columns:

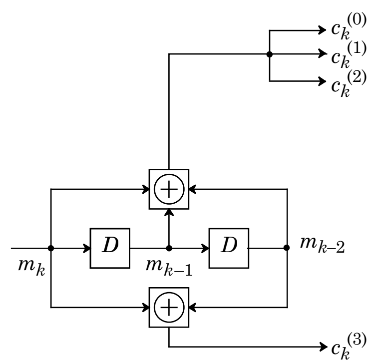
$$\begin{aligned}\mathbf{G}(D) &= [1, D, 1 + D, D^2, 1 + D^2, \ldots, 1 + D + D^2 \ldots + D^{\mu}] \\ &= [10101\ldots1010] + [0110011\ldots0011]D + [0000\ldots01111\ldots1]D^{\mu} \\ &= [1, D, D^2, \ldots D^{\mu}]\begin{bmatrix} 10101010101\ldots01010101 \\ 01100110011\ldots00110011 \\ 00011110000\ldots00001111 \\ \vdots \\ 11111111111\ldots11111111 \\ 00000000000\ldots11111111 \end{bmatrix} \\ &= [1, D, D^2, \ldots D^{\mu}]\mathbf{H}\,, \end{aligned} \tag{S.633}$$

where we recognize $\mathbf{H}$ as the *parity-check* matrix for a ($n = 2^{\mu+1} - 1$, $k = 2^{\mu+1} - \mu - 2$) Hamming code. Among other things, this implies that rows of $\mathbf{H}$ are linearly independent. Therefore, the k-th output block $\boldsymbol{y}_k$ (with dimension $1 \times n$) of this convolutional encoder will be a linear combination of the rows of $\mathbf{H}$. In other words, it will be in the *dual space* to the (n,

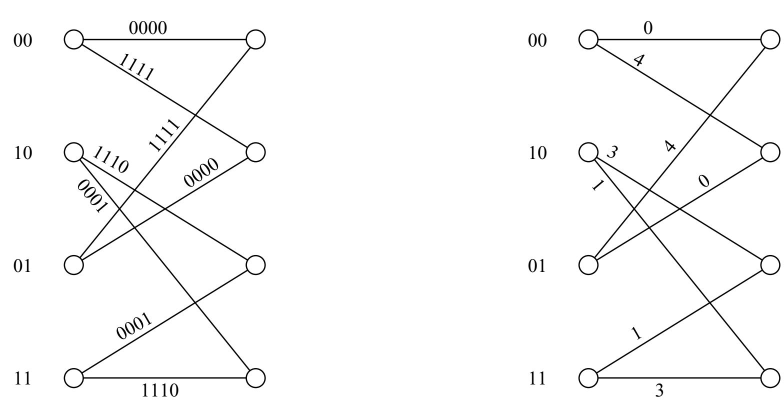
k) Hamming code. Clearly, the way the rows of $\mathbf{H}$ were constructed implies that they all have the same Hamming weight, namely 2^{μ}. In fact, any nonzero linear combination of these rows will also have a Hamming weight of 2^{μ}. (This is a well-known property: the nonzero vectors in the dual space of a Hamming code have the same weight.)

The only way to get an output block of zero is if all of the memory elements are zero; this is a consequence of the linear independence of the rows of $\mathbf{H}$. In other words, the only state transition that produces an all-zero block is the self loop from state zero back to state zero. Every one of the other transitions in the encoder state diagram produce an output block of Hamming weight 2^{μ}. This fact implies that the smallest weight convolutional codeword will be due to the impulsive message $m_i = \delta_i$. In particular, it will produce $\mu + 1$ nonzero output blocks, all of weight 2^{μ}. Therefore, the minimum distance of this convolutional code is $d_{\min} = (\mu + 1)2^{\mu}$.

Problem 12-15. The encoder $\mathbf{G}(D) = [1 + D + D^2, 1 + D + D^2, 1 + D + D^2, 1 + D^2]$ is as sketched below:

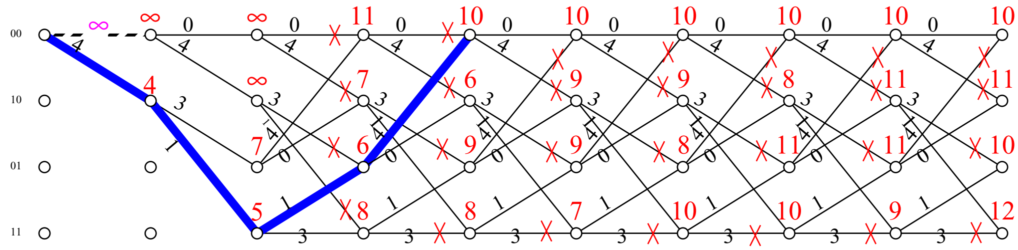


It has two memory elements, and thus four states. One stage of the trellis is shown below:



On the left, the transitions are labeled by the output block $[c_k^{(0)}\ c_k^{(1)}\ c_k^{(2)}\ c_k^{(3)}]$. On the right, the labels are the Hamming weight of the output block.

To find the minimum distance, we can apply the Viterbi algorithm to the following semi-infinite trellis:



The metric for the initial branch from state zero to state zero at stage zero is set to infinity. All remaining branch metrics are the Hamming weight of the corresponding output block. The goal is to find the path back to state zero with the smallest path metric. We can abort the Viterbi algorithm after stage 8, since at that point all of the survivors for the nonzero states have a *larger* survivor metric than the survivor metric for the zero state. Therefore, the highlighted path identifies the minimum-Hamming-weight codeword. The minimum distance is $d_{\text{min}} = 10$. Interestingly, the minimum-weight codeword is produced by a message [11000…], not the impulsive message. (The impulsive message produces a codeword with Hamming weight 4 + 3 + 4 = 11.)

Problem 12-16. In Fig. S-10 we show the state transition diagram with the zero state broken and the branches labeled with z raised to the square of the Euclidean distance of that branch from the zero branch. The path enumerator polynomial is therefore

$$T(z) = z^{12a^2} + z^{16a^2} + \ldots \tag{S.634}$$

In Problem 12-10 we found that the error event with the minimum Euclidean distance had distance $2a\sqrt{3} = \sqrt{12a^2}$, consistent with this result. The compact form of this polynomial is

$$T(z) = \frac{z^{12a^2}}{1 - z^{4a^2}} \ . \tag{S.635}$$

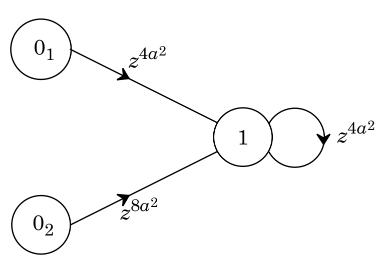


Fig. S-10. A state transition diagram with the zero state broken and the branches labeled with z raised to the square of the Euclidean distance of that branch from the zero branch.

Problem 12-17.

(a) By inspection,

$$T(x, y, z) = x^2yz^3 + x^3y^2z^4 + x^4y^3z^5 + x^5y^4z^6 + \ldots \tag{S.636}$$

or equivalently

$$T(x, y, z) = \frac{x^2yz^3}{1 - xyz} \ . \tag{S.637}$$

(b) From (S.636) the length four error event (corresponding to x^5) has four bit errors and Hamming distance 6.

(c) The broken state transition diagram of Fig. 12-35 can be modified as shown in Fig. S-11. Since we are only interested in length and distance (and not the number of bit errors) we need only two variables, x and z. The path enumerator polynomial is found using (12.185):

$$T(x, z) = \frac{x^3z^5}{1 - x^2z - xz} \ . \tag{S.638}$$

By long division, we compute the first few terms

$$T(x, z) = x^3z^5 + x^4z^6 + x^5z^6 + x^5z^7 + x^6z^8 + x^7z^7 + \ldots \tag{S.639}$$

From this we see that there are two distance 6 error events, one with length 3 and one with length 4. (The length is the exponent of x minus one.) Also, there are two length four error events, one with distance 6 and one with distance 7.

Problem 12-18. The generator for the (3, 1) repetition code is $\mathbf{G} = [1, 1, 1]$. It spans a one-dimensional subspace. The set of vectors orthogonal to this subspace is $\{110, 011, 101, 000\}$.

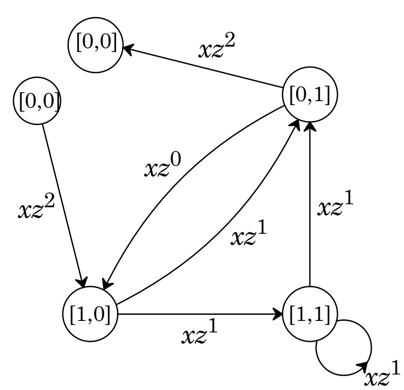
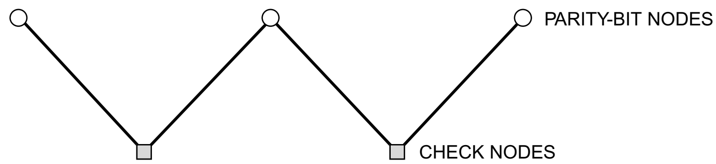


Fig. S-11. The state transition diagram of Fig. 12-35 is modified for enumerating the path lengths as shown.

Any two of the nonzero vectors can be stacked to produce a valid parity-check matrix. For example, taking the first two yields a parity-check matrix of:

$$\mathbf{H} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}. \tag{S.640}$$

(a) The corresponding Taner graph is:



Clearly it has no cycles.

(b) Suppose that a message bit of $m = 0$ gets mapped to the symbols $a_0 = a_1 = a_2 = -1$, while $m = 1$ gets mapped to $a_0 = a_1 = a_2 = +1$. The channel adds independent noise $\mathcal{N}(0, \sigma^2)$, producing $\boldsymbol{r} = [r_0, r_1, r_2]^T$. First we find the exact a posteriori LLR for the message bit, defined by (12.106):

$$\begin{aligned} \lambda &= \log \frac{\Pr[m = 1 \mid \boldsymbol{r}]}{\Pr[m = 0 \mid \boldsymbol{r}]} \\ &= \log \frac{f(\boldsymbol{r} \mid m = 1)\Pr[m = 1]}{f(\boldsymbol{r} \mid m = 0)\Pr[m = 0]} \\ &= \log \frac{f(\boldsymbol{r} \mid m = 1)}{f(\boldsymbol{r} \mid m = 0)}, \end{aligned} \tag{S.641}$$

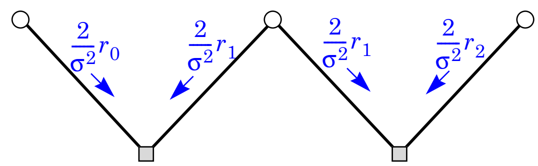
where the second equality follows from Bayes' rule, and the last equality follows from our assumption that the message bit is equally likely to be zero and one. In AWGN, this reduces to:

$$\begin{aligned} \lambda &= \log \frac{f(\boldsymbol{r} \mid m = 1)}{f(\boldsymbol{r} \mid m = 0)} \\ &= \log \left(\frac{(2\pi\sigma^2)^{-3/2} e^{\frac{-1}{2\sigma^2}\|\boldsymbol{r} - \boldsymbol{s}\|^2}}{(2\pi\sigma^2)^{-3/2} e^{\frac{-1}{2\sigma^2}\|\boldsymbol{r} + \boldsymbol{s}\|^2}} \right) \qquad \text{(where we introduce } \boldsymbol{s} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}) \\ &= \frac{1}{2\sigma^2} \left(\| \boldsymbol{r} + \boldsymbol{s} \|^2 - \| \boldsymbol{r} - \boldsymbol{s} \|^2 \right) \\ &= \frac{1}{2\sigma^2} \left(2\boldsymbol{s}^T\boldsymbol{r} - (-2\boldsymbol{s}^T\boldsymbol{r}) \right) \\ &= \frac{2}{\sigma^2} \boldsymbol{s}^T\boldsymbol{r} \\ &= \frac{2}{\sigma^2} \sum_i r_i \, . \end{aligned} \tag{S.642}$$

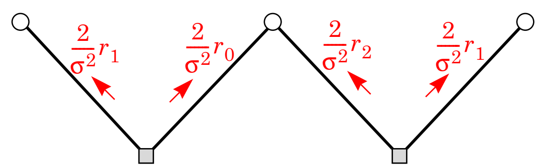
This is the exact LLR for the message bit. (It is also the exact LLR for each of the three coded bits, since the three coded bits are identical for the repetition code.)

Now let's look at the LLR's produced by message passing.

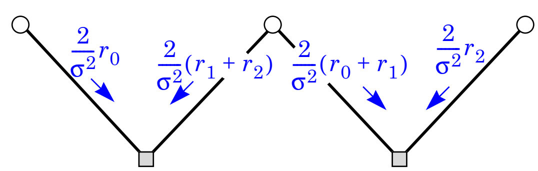
- The initial downward message from the i-th bit node is $\frac{2}{\sigma^2} r_i$:



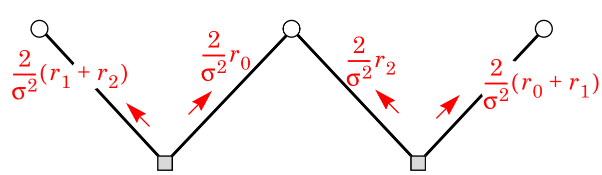
- The tanh rule reduces to a "pass through" rule for check nodes of degree two, so that the upward messages for the first iteration are as shown below:



- The first and last bit node receive only one upward message. Their downward message on the same branch must exclude this message, so their downward messageis the same as before, namely, $(2/\sigma^2)r_i$. The middle bit node, on the other hand, receives two upward messages. According to the message-passing algorithm, the downward messages of the second iteration are as shown below:



- Again using the "pass through" rule, the upward messages for thesecond iteration are:



After the second iteration, the i-th bit node produces its estimate of λ_i by adding its channel value $\frac{2}{\sigma^2} r_i$ to its incoming upward messages, yielding:

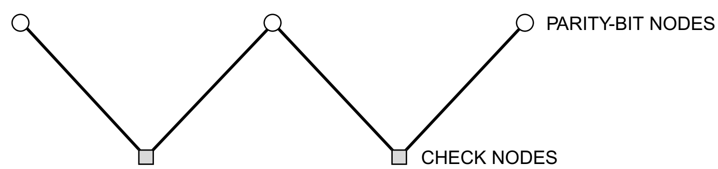
$$\lambda_0 = \frac{2}{\sigma^2} r_0 + \frac{2}{\sigma^2} (r_1 + r_2)$$
$$\lambda_1 = \frac{2}{\sigma^2} r_1 + \frac{2}{\sigma^2} r_0 + \frac{2}{\sigma^2} r_2$$
$$\lambda_2 = \frac{2}{\sigma^2} r_2 + \frac{2}{\sigma^2} (r_0 + r_1), \tag{S.643}$$

for bit nodes 0, 1, and 2, respectively. We see that all three are identical, as they should be for the repetition code, and that all three reduce to the exact LLR of $\frac{2}{\sigma^2} \sum_i r_i$, as found in (S.642).

Problem 12-18. Yes. When $m = 2$, the Hamming code reduces to the $(3,\ 1)$ repetition code. A valid parity check matrix for the repetition code is:

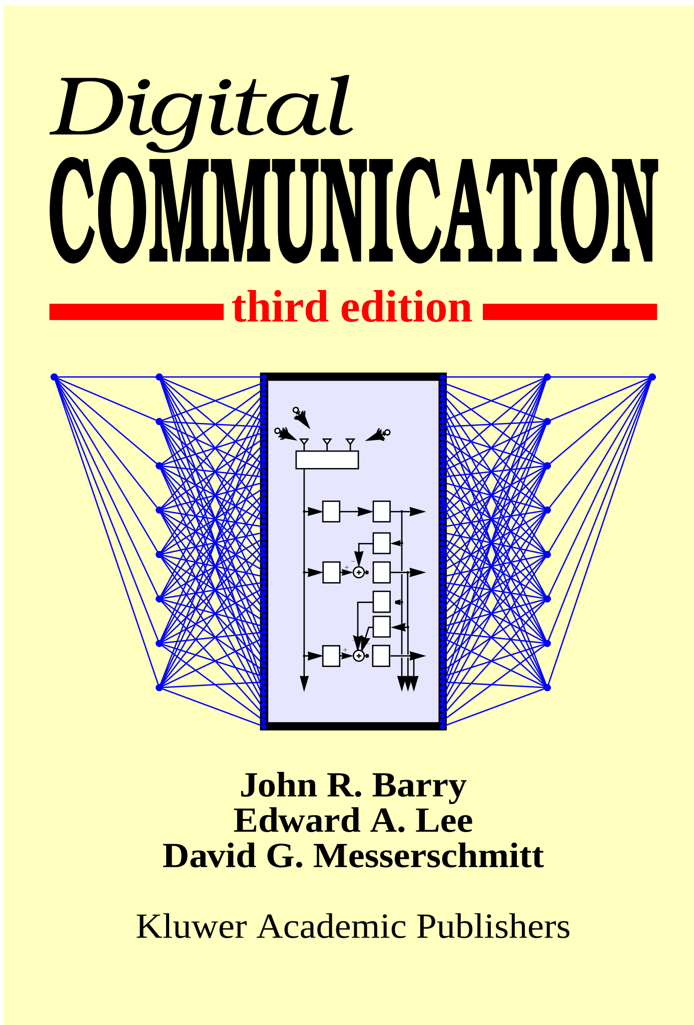
$$\mathbf{H} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \tag{S.644}$$

The corresponding Taner graph is obviously cycle-free:



Solutions Manual

for



Chapter 13 – Signal Space Coding

Problem 13-1. In the transmitter, generate two-dimensional symbols in the conventional way. Now group the transmitted baseband symbols into pairs, and transmit the real part of the two-dimensional symbol first, then the imaginary part. In the receiver, group the received samples at the slicer into pairs, and apply them to a complex slicer.

Problem 13-2.

(a) The coding gain is, of course, $\gamma_\Lambda = 1$. The shaping gain is

$$\gamma_S = \frac{2\pi R^2}{12R^2/2} = \frac{\pi}{3} \ . \tag{S.641}$$

(b) The coding gains are the same. The circular shaping provides $10 log_{10}\pi/3 = 0.2$ dB of shaping gain.

Problem 13-3. For the hexagonal constellation, the fundamental volume is the volume of a hexagon with inscribed radius $d_{\min}/2$. Hence,

$$V(\Lambda) = 6(d_{\min}/2)^2 \tan(\pi/6) \tag{S.642}$$

and the coding gain is

$$\gamma_\Lambda = \frac{2}{3\tan\left(\frac{\pi}{6}\right)} = 1.155 \ . \tag{S.643}$$

This is 0.6 dB.

Problem 13-4. Clearly

$$d_{\min}(\alpha \cdot \Lambda) = \alpha \cdot d_{\min}(\Lambda) \tag{S.644}$$

and

$$V(\alpha \cdot \Lambda) = \alpha^N \cdot V(\Lambda) \ . \tag{S.645}$$

Thus, as a function of α, the coding gain is

$$\gamma_{\alpha \cdot \Lambda} = \frac{\alpha^2 d_{\min}^2}{\alpha^2 V^{2/N}(\Lambda)} = \frac{d_{\min}^2}{V^{2/N}(\Lambda)} = \gamma_\Lambda \ . \tag{S.646}$$

Problem 13-5. The volume is

$$V[C_N(R)] = \int_{C_N(R)} d\boldsymbol{x} \ . \tag{S.647}$$

The integral can be separated into the product of N integrals, each evaluating to $2R$, so $V[C_N(R)] = (2R)^N$. To evaluate the power, the uniform density function over the N-cube has height $(2R)^{-N}$, so the power is

$$P[C_N(R)] = (2R)^{-N} \int_{C_N(R)} \| \boldsymbol{x} \|^2 d\boldsymbol{x} = (2R)^{-N} \sum_{i=1}^{N} \int_{C_N(R)} x_i^2 d\boldsymbol{x} \,. \qquad \text{(S.648)}$$

Each of the integrals evaluates to the same value,

$$\int_{C_N(R)} x_i^2 d\boldsymbol{x} = V[C_{N-1}(R)] \frac{2R^3}{3} \;, \qquad \text{(S.649)}$$

and thus

$$P[C_N(R)] = N \cdot R^2 / 3 \;. \qquad \text{(S.650)}$$

Substituting the volume and power into the shaping gain, we get $\gamma_{C_N(R)} = 1$.

Problem 13-6.

(a) Assuming the radius of the N-sphere is R, P is the power of $S_N(R)$ divided by the number of complex symbols, $N/2$. Hence $P = 2R^2/(N+2)$, and for fixed P the radius as a function of N is $R^2 = (N+2)P/2$. Assuming N is even, the volume of a sphere of this radius is

$$V\left(S_N\left(\sqrt{\left(\frac{N}{2}+1\right)P}\right)\right) = \frac{\left(\pi\left(\frac{N}{2}+1\right)P\right)^{N/2}}{\left(\frac{N}{2}\right)!} \;. \qquad \text{(S.651)}$$

The number of points in the signal constellation is proportional to the volume, since the fundamental volume is assumed to be constant, and hence ν is proportional to the logarithm of the volume divided by $N/2$, or

$$\nu \quad \alpha \quad \frac{2}{N} \cdot \log_2 \frac{\left(\pi\left(\frac{N}{2}+1\right)P\right)^{N/2}}{\left(\frac{N}{2}\right)!} = \log_2(\pi P) + \log_2\left(\frac{N}{2}+1\right) - \frac{2}{N} \cdot \log_2(N/2)! \;. \qquad \text{(S.652)}$$

Thus, the spectral efficiency is a constant plus a term that depends on N, where the latter is

$$\log_2\left(\frac{N}{2}+1\right) - \frac{2}{N} \cdot \log_2(N/2)! \,. \qquad \text{(S.653)}$$

When N goes from 2 to 4, the spectral efficiency increases by 0.085 bits. When N goes from 2 to 6, the spectral efficiency increases by 0.138 bits.

(b) By the Sterling approximation, the increase in spectral efficiency as $N \to \infty$ relative to $N = 2$ is $\log_2 e - 1 = 0.443$ bits per complex symbol. Of course, the absolute (as opposed to relative) spectral efficiency depends on Λ as well as N.

Problem 13-7. Let us assume that R is chosen such that $\boldsymbol{X}_N$ has unit variance components; that is, $R^2 = N + 2$. The marginal density of $\boldsymbol{X}_K$ will be the density of $\boldsymbol{X}_N$ integrated over $N - K$ components. The integral is over a sphere of radius $\sqrt{R^2 - \|\boldsymbol{x}_K\|^2}$ and dimension $N - K$. The integrand is the density of $\boldsymbol{X}_N$, which is a constant $1 / V_N(R)$, and thus the integral becomes proportional to the volume of an $(N - K)$-dimensional sphere. Thus,

$$f_{bXK}(bx_K) = \{\frac{V_{N-K}(\sqrt{R^2 - \|\boldsymbol{x}_K\|^2})}{V_N(R)} = \frac{V_{N-K}(1)}{V_N(1)} \frac{\left(1 - \frac{\|\boldsymbol{x}_K\|^2}{R^2}\right)^{N/2}}{\left(1 - \frac{\|\boldsymbol{x}_K\|^2}{R^2}\right)^{K/2}} . \tag{S.654}$$

Taking $R^2 = N + 2$, or equivalently $R^2 = N$ when N is large, the only term that is a function of $\boldsymbol{x}_K$ as $N \to \infty$ is the numerator, which assumes the functional form:

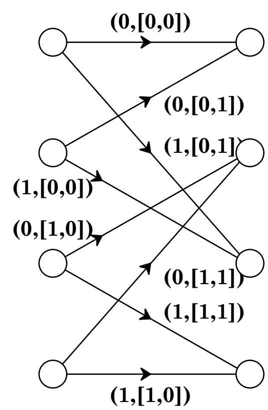
$$\exp\{-\|\boldsymbol{x}_K\|^2 / 2\} , \tag{S.655}$$

a Gaussian density with unit variance and independent components. The remaining constants are of course the normalization to unit area, and must equal $(2\pi)^{-K/2}$ asymptotically.

Problem 13-8. It is straightforward to calculate this probability,

$$\Pr\{\| \boldsymbol{X} \| \le R - \varepsilon\} = \frac{V(S_N(R - \varepsilon))}{V(S_N(R))} = \frac{(R - \varepsilon)^N}{R^N} = \left(1 - \frac{\varepsilon}{R}\right)^N \to \quad 0 . \tag{S.656}$$

Problem 13-9. The trellis for the convolutional coder is shown in the following figure:

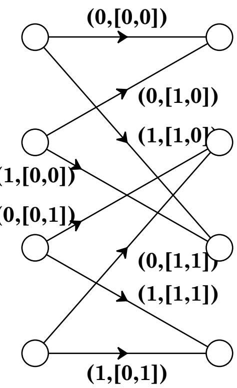


Comparing this to (XREF FIG 14.9)a, if we choose the mapping:

$C_k^{(1)} C_k^{(2)}$	A_k
0 0	a
0 1	$-a$
1 0	ja
1 1	$-ja$

then the trellis is identical. Thus with a line coder that implements this mapping, the code is equivalent.

Problem 13-10. The trellis for the convolutional coder is shown in the following figure:



To get a trellis equivalent to that in (XREF FIG 14.9) we select the mapping:

$C_k^{(1)}C_k^{(2)}$	A_k
0 0	a
0 1	ja
1 0	$-a$
1 1	$-ja$

Problem 13-11. One stage of the trellis is shown in Fig. S-12a. The minimum-distance error event is shown in Fig. S-12b for three different correct state trajectories. Also shown is the distance. The distance is different for all three.

Problem 13-12.

(a) Four bits per symbol are required, so the alphabet needs 16 symbols. 16-QAM will work.

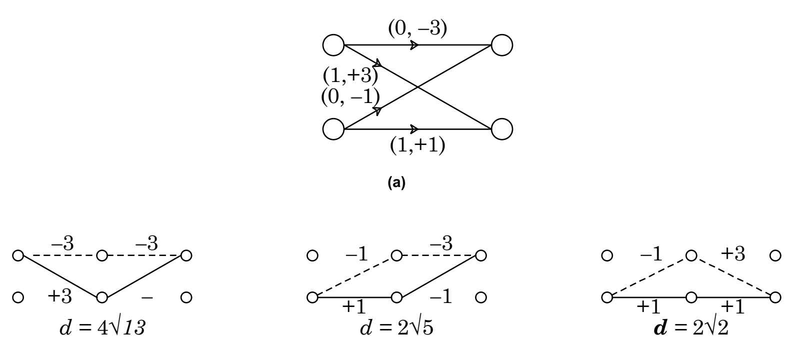


Fig. S-12. a. One stage of the trellis for the trellis coder in (XREF 14.33). b. The minimum-distance error events when the correct state trajectory is the path shown as a dashed line.

(b) The required alphabet size is 32. The cross constellation in (XREF FIG 14.23) will work, although there are others in Chapter 6.

(c) Coding is required. To get 4 dB total gain, using Ungerboeck's rule of thumb (see the first paragraph in Section 14.2.1), an eight state trellis code will work. The coder in (XREF FIG 14.23) will do the job, and provide the additional benefit of 90 degree phase invariance. If phase invariance is not an issue, then the coder in (XREF FIG 14.18) will also work, although one additional uncoded bit is required.

Problem 13-13. Consider the error events in (XREF FIG 14.38). There are several error events represented here because of the parallel paths. To find the minimum-distance error event of these, first find the minimum distances between transitions in the same stage of the trellis. These are shown in the figure. For example, in the first stage, the upper parallel pair of transitions are taken from subset A in (XREF FIG 14.13). The lower pair are taken from subset C. Hence the minimum distance in this stage is the minimum distance between symbols in A and C, or √2. The minimum distance in the third stage is similarly computed. The minimum distance in the second stage is the minimum distance between symbols in A and B, which with some simple geometry can be shown to be √{2 –√2} ≈ 0.77. The minimum distance of all the error events shown is the square root of the sums of the squares of these stage distances, or

$$\sqrt{\{2+2-\sqrt{2}+2\}} = \sqrt{\{6-\sqrt{2}\}} \approx 2.14 \ . \tag{S.657}$$

It is easy to see that any error event with length greater than three stages will have a distance greater than these error events, so 2.14 is the distance of the second closest error event.

Problem 13-14.

(a) The partition is shown in (XREF FIG 14.39), along with the minimum distances. Notice that at the final partitioning stage (into 16 subsets) there is no improvement in minimum distance for some of the subsets.

(b) The average power of the 16-QAM constellation has been computed elsewhere and is 10. The 32-cross constellation has all the same points, plus 16 additional points with average power

$$\frac{8}{16}(5^2+3^2) + \frac{8}{16}(5^2+1^2) = 30 \tag{S.658}$$

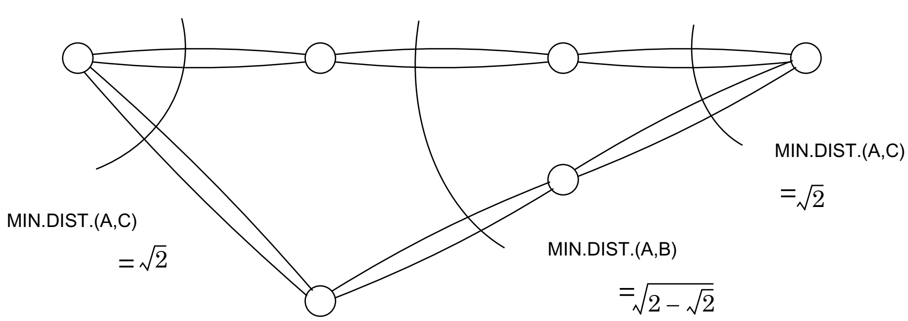


Fig. S-13. A set of error events and their minimum distance.

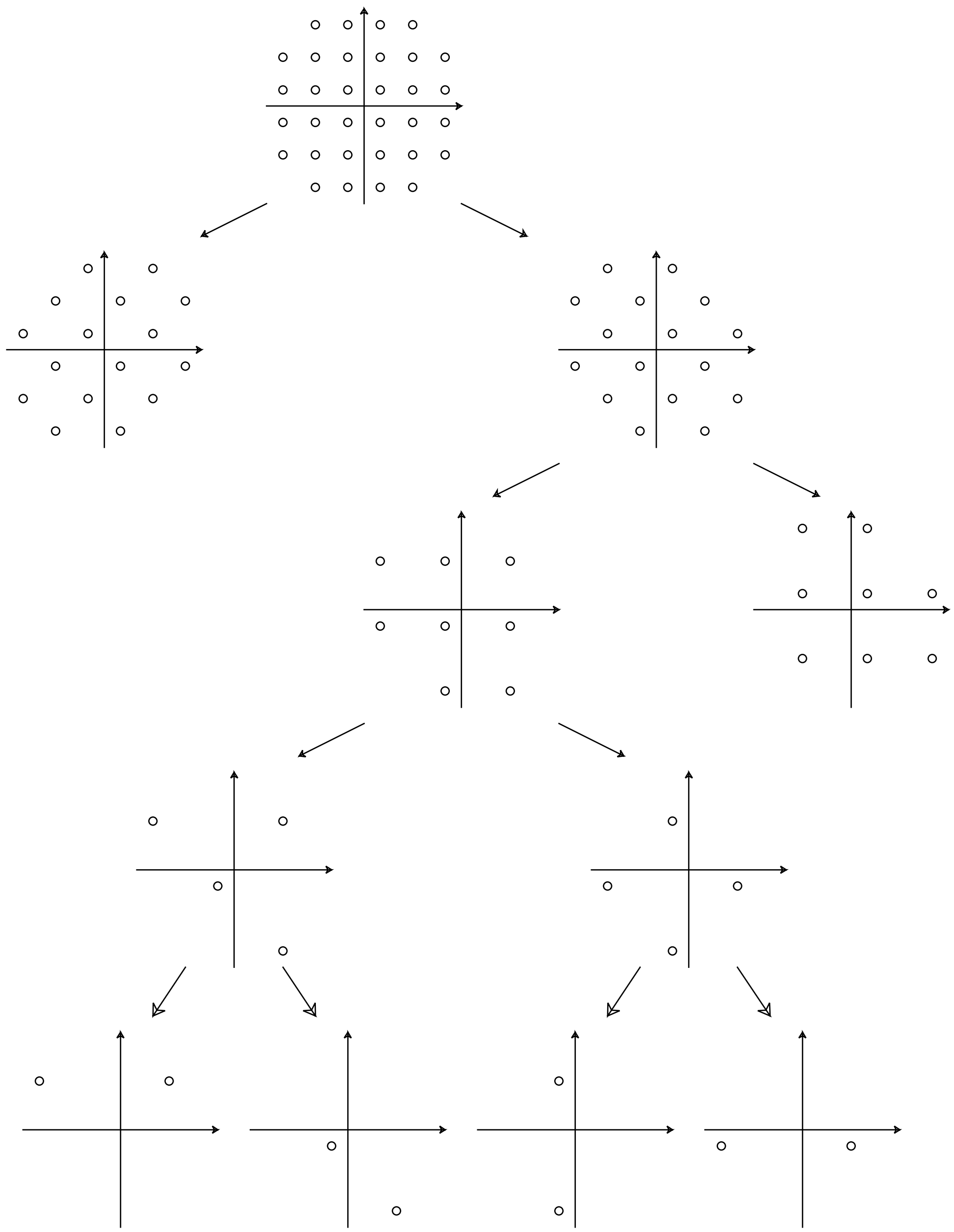


Fig. S-14. Set partitioning for a 32-cross constellation. The minimum distances between points in the subset are shown.

so the overall average power is

$$\frac{1}{2}(10 + 30) = 20 \quad . \tag{S.659}$$

This is $10\log(2) = 3$ dB more power.

(c) It should be adequate to use the subsets in the third row of (XREF FIG 14.39). The minimum distance between parallel transitions is 4. There are 4 such subsets, so $mtilde$ = 1. Compared to the 16-QAM constellation in part (b), which has minimum distance 2, this is a 6 dB improvement. However, the power of the 32-cross constellation has to be reduced by 3 dB, so if the parallel transitions have the smallest distance (we can assure this with proper coder design) then the total gain will be about 3 dB. Using Ungerboeck's rule of thumb (see the first paragraph of Section 14.2.1), a coder with 4 states should work.

(d) It should be adequate to use the subsets in the fourth row of (XREF FIG 14.39). There are 8 such subsets, so $\tilde{m} = 2$. The minimum distance between parallel transitions will be $4\sqrt{2}$, which is about 9 dB better than the minimum distance of 2 in the 16-QAM constellation. Again, of this 9 dB improvement, 3 dB must be sacrificed to normalize the power, leaving a 6 dB gain. This is more than we need, so we could use a 16 state trellis coder to get about 5 dB gain, and the minimum-distance error event will probably not be the parallel transitions.

Problem 13-15.

(a) The trick here is to compare to an uncoded system with the same average power. If the 8-PSK symbols have amplitude a, then it has average power a^2. If the 16-QAM symbols have real and imaginary parts that are $\pm\ b$ or $\pm\ 3b$, then they have average power $10b^2$. Hence, for the two systems to have the same average power, we require that $a = \sqrt{10}b \approx 3.16b$. The subsets that are used for parallel transitions are in the third row of (XREF FIG 14.16). The symbols within each subset have minimum distance $4b$, so if the parallel transitions dominate the probability of error, then

$$\Pr[error] \approx Q\left(\frac{2b}{\sigma}\right) \quad . \tag{S.660}$$

The minimum distance for the 8-PSK alphabet is $a\sqrt{2 - \sqrt{2}} \approx 0.77a$, so for the uncoded system

$$\Pr[error] \approx Q\left(0.77\frac{a}{2\sigma}\right) \quad . \tag{S.661}$$

The difference is

$$20\log\left(\frac{4b}{0.77a}\right) = 20\log\left(\frac{4}{0.77\sqrt{10}}\right) = 4.3 \text{ dB}, \tag{S.662}$$

which is very good indeed.

(b) The trellis is shown in (XREF FIG 14.40). We can show that the error event with shape as in (XREF FIG 14.38) is $\sqrt{20}\,b$, which is greater than the distance $4b$ of the parallel transitions. Hence the assumption in (a) seems reasonable.

Problem 13-16. First note that every error event is a minimum-distance error event for all possible actual paths through the trellis. Note further that every error event starts and ends with a symbol error, and has no symbol errors in between, so $w(e) = 2$ for all $e \in E$. Consequently, (XREF EQN 9.150) becomes

$$R = 2\Sigma_{e\in\, E}\Pr[\psi] \quad . \tag{S.663}$$

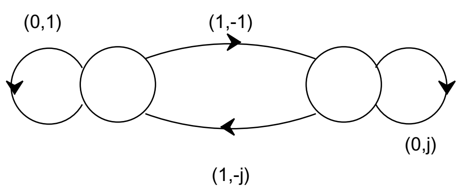
Define $E(\psi)$ to be the set of error events for the actual path ψ through the trellis. Then the summation in XREF LAST can be rewritten

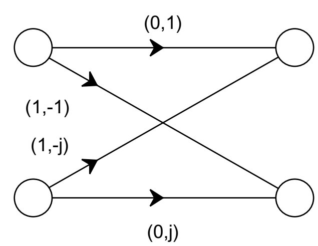
$$\Sigma_{e\in\, E}\Pr[\psi] = \Sigma_{\psi}\Sigma_{e\in\, E(\psi)}\Pr[\psi] = (\Sigma_{\psi}\Pr[\psi])(\Sigma_{e\in\, E(\psi)}1) \quad . \tag{S.664}$$

The first summation is unity, and the second is infinite because every ψ has an infinite number of error events. Consequently, R is unbounded.

Problem 13-17.

(a) One stage of the trellis is shown in the following figure.





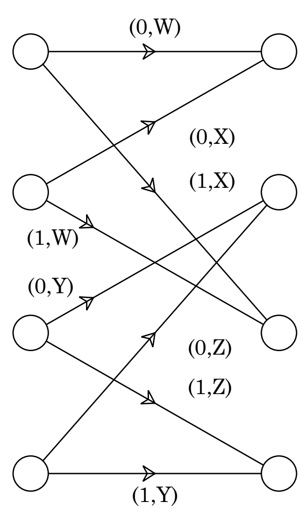
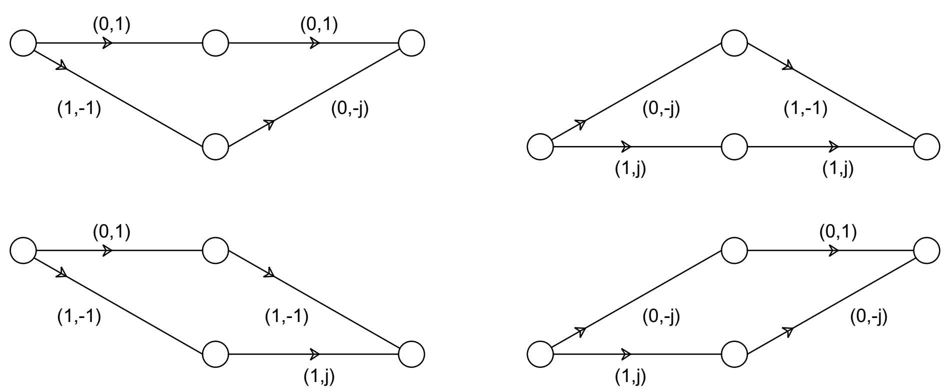


Fig. S-15. Yet another trellis.

(b) For different assumed correct state trajectories, the minimum-distance error events are shown below.



In each case the distance is $d_{E,\min} = \sqrt{6}$. The probability of occurrence is therefore $Q(\sqrt{6} / 2\sigma)$.

(c) The 2-PSK alphabet with the same power is $\mathcal{A} = \{-1, +1\}$ which has a minimum distance of 2. The coded system is therefore

$$20\log(\sqrt{6} / 2) = 1.76\ dB \tag{S.665}$$

better.

Errata

Last updated: April 5, 2007

Are there more? Please send feedback to barry@ece.gatech.edu. Thank you.

page 16: The second $-1/2T$ in Fig. 2-4(b) should be $1/2T$.

page 17: The Hilbert transformer gives a $-\pi/2$ phase shift, not a $-\pi$ phase shift.

page 44: Remove the subscript e on the right-hand side of the equation.

page 47: Correct table 2-3: $\mathrm{rect}(t, T/2) \leftrightarrow \frac{\sin(\pi f T)}{\pi f T}$, and $\frac{\sin(Wk)}{WkT} \leftrightarrow \mathrm{rect}(\theta, W)$.

page 60: In equation (3.15), exchange the order of $d\alpha$ and $d\beta$.

page 60: After (3.21), replace “a complex variable s” by “an imaginary variable $s = j\omega$”; replace “Laplace” by “Fourier”; and replace “at $-s$” by “at $-\omega$”.

page 73: Third line of (3.79) is missing $d\theta$.

page 81, last line, a “j” is missing: replace “$H(f) = -\mathrm{sign}(f)$” by “$H(f) = -j \cdot \mathrm{sign}(f)$”.

page 82, (3.109): the minus sign before $R_{\hat{N}N}(-\tau)$ should be a plus sign.

page 83, second line after (3.114): “X_{k+1}” should be “X_k”.

page 92, (3.133): the minus sign before $\lambda q_j(t)$ should be a plus sign.

page 94, (3.144): $p_j(t)$ should be $p^j(t)$.

page 99: The first convolution in (3.163) is missing a factor of s.

page 100, (3.172): there is a factor of “$H(f)$” missing from the right-hand side.

page 103, (3.188): the first summation $\sum_{m=0}^{\infty}$ should be $\sum_{m=-\infty}^{\infty}$.

page 104, (3.194): The factor of $x(t_0)$ is missing a minus sign: it should be $e^{-\Lambda(t)}$.

page 104, (3.196): Replace $e^{-\Lambda(u)}$ inside the integral by $e^{\Lambda(u)}$.

page 107, Problem 3-9: remove the absolute value in $f_a < |f| < f_b$.

page 108, (3.210): Replace a by q.

page 120, (4.21): Remove the lower bound “$0 \le$”.

page 121, the right-hand side of (4.23) is missing a minus sign.

page 122: The axis labels at the bottom of Fig. 4-4 were cut off during production; they should be $\{-10, -5, 0, 5, 10, \ldots, 35, 40\}$, as shown in the figure at the end of this document.

page 123, (4.25) is missing a minus sign and a dx.

page 132: The summation $\sum_m$ in (5.1) should be $\sum_k$.

page 139, $\{a_l \ne k\}$ should be $\{a_{l \ne k}\}$.

page 149, the middle term of the second line of (5.33) should be $\mathrm{Re}\{(m-n)^*(E[a]-m)\}$.

page 150, the second line of (5.34) is missing a factor of $1/L$.

page 159: The factor of K in the third line after (5.55) should be conjugated.

page 160, Exercise 5-1, second line, $p(t)$ should be $h(t) * h^*(-t)$.

page 162, in the second line of the second paragraph, “filter” should be “sampler”.

page 171, the convolution near the end of the paragraph after (5.77) should be $a_k * m_k$.

page 188, Exercise 5-35 should refer to Fig. 5-34, not Fig. 5-32.

page 189: In the last line of Example 5-36, "4-PAM" should be "4-PSK".

page 190: In (5.113), the second factor of $\left(1-\frac{1}{\sqrt{M}}\right)$ should be squared, yielding:

$$\Pr[\text{error}] = 4\left(1-\frac{1}{\sqrt{M}}\right)Q\left(\sqrt{\frac{3E/N_0}{M-1}}\right) - 4\left(1-\frac{1}{\sqrt{M}}\right)^2 Q^2\left(\sqrt{\frac{3E/N_0}{M-1}}\right). \tag{E.1}$$

page 194: The label in Fig. 5-38(a) should be 8.4 dB, to match that in part (b).

page 204: After (6.5), insert "assuming that $f_c T$ is an integer".

page 211: The second element of $\boldsymbol{h}_i$ after (6.25) should be $h_{i,2}$.

page 225: The x-axis label of Fig. 6-11 should be $1/\nu$, not $1/b$.

page 235: The center frequency in the top figure should be $15/T$, not $15/2T$.

page 245: The r.h.s. of (6.104) should be $1/N$, as should $|x_m|^2$ at top of p. 245.

page 318: In (7.85) and (7.86), both of the ">" should be "<", so that $\hat{a}_0 = \hat{a}_1 = \hat{a}_2 = 0$.

page 334: Replace in_k by X_k

page 343: In (7.169) V_l should be V_m.

page 532, Prob. 10-4: a factor of e^{-t} is missing from both impulse responses; they should be $h_1(t) = e^{-t}u(t-T)$ and $h_2(t) = e^{-t}u(t-2T)$.

page 532, Prob. 10-5: "single-input double-output" should be "double-input single-output"